Approximating Domination on Intersection Graphs of Paths on a Grid

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Abstract. A graph G is called B_k*-EPG* (resp., B_k*-VPG*), for some constant $k \geq 0$, if it has a string representation on an axis-parallel grid such that each vertex is a path with at most k bends and two vertices are adjacent in G if and only if the corresponding strings share at least one grid edge (resp., the corresponding strings intersect each other). If two adjacent strings of a B_k-VPG graph intersect each other exactly once, then the graph is called a *one-string* B_k-VPG graph.
In this paper, we study the Minimum Dominating Set problem on B_1-EPG and B_1-VPG graphs. We first give an $O(1)$-approximation algorithm on one-string B_1-VPG graphs, providing the first constant-factor approximation algorithm for this problem. Moreover, we show that the Minimum Dominating Set problem is APX-hard on B_1-EPG graphs, ruling out the possibility of a PTAS unless P=NP. Finally, to complement our APX-hardness result, we give constant-factor approximation algorithms for the Minimum Dominating Set problem on two non-trivial subclasses of B_1-EPG graphs.

1 Introduction

In this paper, we study the Minimum Dominating Set problem on B_1-VPG and B_1-EPG graphs. These are two special subclasses of *string graphs*, which are of interest in several applications such as circuit layout design and bioinformatics. A graph is called a B_k*-EPG graph* if it has an EPG representation (stands for Edge representation of Paths in a Grid) in which each path has at most k bends. In this paper, we are interested in B_k-EPG graphs for $k = 1$.

Definition 1 (B_1-EPG Graph). *A graph $G = (V, E)$ is called a* B_1-EPG graph*, if every vertex u of G can be represented as a* path P_u *on a grid $\mathcal{G}$ such that (i) P_u has at most one bend, and (ii) paths $P_u, P_v \in P$ share a grid edge of $\mathcal{G}$ if and only if $(u, v) \in E$.*

Similarly, a graph is said to have a VPG representation (stands for Vertex representation of Paths in a Grid), if its vertices can be represented as simple paths on an axis-parallel grid such that two vertices are adjacent if and only if the corresponding paths share at least one grid node. Although these graphs were considered a while ago when studying string graphs [15], they were formally

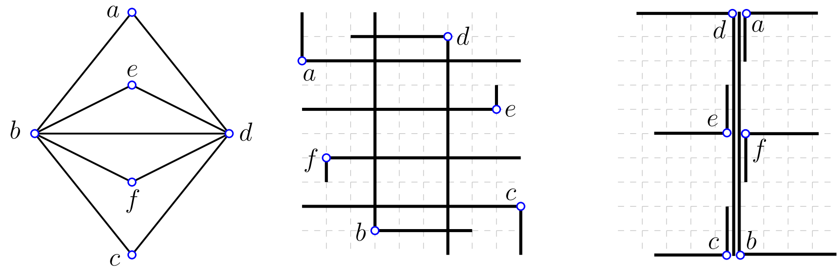


Fig. 1: A graph on six vertices (left) with its B_1-VPG (middle) and B_1-EPG (right) representations. Notice that the vertices e and f are not adjacent in the B_1-EPG representation as they only share a grid node (but not a grid edge).

investigated by Asinowski et al. [2]. Similar to B_k-EPG graphs, a *B_k-VPG graph* is a VPG graph in which each path has at most k bends. In this paper, we are interested in B_k-VPG graphs for $k = 1$.

Definition 2 (B_1-VPG Graph). *A graph $G = (V, E)$ is called a* B_1-VPG graph*, if every vertex u of G can be represented as a* path P_u *on a grid $\mathcal{G}$ such that (i) P_u has at most one bend, and (ii) two paths P_u and P_v intersect each other at a grid node if and only if $(u, v) \in E$.*

We remark that by *intersecting* each other, we exclude the case where two paths only *touch* each other; that is, no endpoints of a path belongs to any other path. Figure 1 shows a graph with its B_1-VPG and B_1-EPG representations. A string graph is called *one-string* if it has a string representation in which curves intersect at most once [6]. By combining one-string and B_1-VPG representations, a *one-string* B_1-VPG *graph* is defined as a B_1-VPG graph in which two paths intersect each other exactly once whenever the corresponding vertices are adjacent.

In this paper, we study the Minimum Dominating Set problem on B_1-EPG and B_1-VPG graphs. Let $G = (V, E)$ be an unweighted, undirected graph. A set $S \subseteq V$ is a dominating set if every vertex in $V \setminus S$ is adjacent to some vertex in S. The objective of the Minimum Dominating Set problem is to compute a dominating set S of minimum size. Minimum Dominating Set is a fundamental optimization problem in graph theory, which arises in many applications such as wireless sensor networks, scheduling and resource allocation; the problem is well known to be NP-hard.

Related work and our results. It is known that every circle graph is a one-string B_1-VPG graph [2]. Since Minimum Dominating Set is APX-hard on circle graphs [8], the problem becomes APX-hard also on one-string B_1-VPG graphs. However, to the best of our knowledge, there is no approximation algorithm known for the problem. We note that there are $O(1)$-approximation algorithms for Minimum Dominating Set on circle graphs [9, 10], but these algorithms do not work for B_1-VPG graphs as they heavily rely on the fact that the vertices of the input graph are modelled as chords of a circle. For B_1-EPG graphs, there exists a 4-approximation algorithm for the Minimum Dominating Set problem [5], as it is known that such graphs are a subclass of 2-interval graphs [12]. However, we were unable to find a reference on the complexity of the Minimum Dominating Set problem on B_1-EPG graphs. In fact, Epstein et al. [11] left open studying the Minimum Dominating Set problems on B_1-EPG graphs.

In this paper, we present the following results:

- We give a polynomial-time $O(1)$-approximation algorithm for the Minimum Dominating Set problem on *one-string* B_1-VPG graphs, providing the *first* constant-factor approximation algorithm for this problem on B_1-VPG graphs.
- We prove that the Minimum Dominating Set problem is APX-hard on B_1-EPG graphs, even if only two types of paths are allowed in the input graph. Thus, there exists no PTAS for this problem on B_1-EPG graphs unless P=NP.
- We give polynomial-time constant-factor approximation algorithms for the Minimum Dominating Set problem on two subclasses of B_1-EPG graphs.

Organization. We first give some notations and definitions in Section 2. We give our $O(1)$-approximation algorithm for the Minimum Dominating Set problem on one-string B_1-VPG graphs in Section 3. Then, we present our results for the Minimum Dominating Set problem on B_1-EPG graphs in Section 4. We conclude the paper with a discussion on open problems in Section 5.

2 Notation and Definitions

For a B_1-VPG (resp., B_1-EPG) graph G, we use $\langle \mathcal{P}_{\texttt{vtx}}, \mathcal{G} \rangle$ (resp., $\langle \mathcal{P}_{\texttt{edg}}, \mathcal{G} \rangle$) to denote a B_1-VPG (resp., B_1-EPG) representation of G, where $\mathcal{P}_{\texttt{vtx}}$ (resp., $\mathcal{P}_{\texttt{edg}}$) is the collection of paths corresponding to the vertices of G and $\mathcal{G}$ is the underlying grid. Since the recognition problem is NP-hard on such graphs [12, 7, 17], we assume throughout this paper, that we are always given a string representation of a B_1-VPG or B_1-EPG graph as input (in addition to G). We sometimes violate the wording and say *path(s) in G* to actually refer to the vertices in G corresponding to the paths in $\mathcal{P}_{\texttt{vtx}}$ or $\mathcal{P}_{\texttt{edg}}$. We denote the x- and y-coordinates of a point p in the plane by $x(p)$ and $y(p)$, respectively.

Let P be a grid path with at most one bend. Since P has at most one bend, it is in one of the types {⌜, ⌞, ⌝, ⌟, |, −}. We call a path P of type x ∈{⌜, ⌞, ⌝, ⌟}, a ***x**-type path*. Similar to [11], we complete the definition by referring to no-bend paths as ⌞-type paths.

We denote the horizontal and vertical segments of P by $\texttt{hPart}(P)$ and $\texttt{vPart}(P)$, respectively. We call the common endpoint of $\texttt{hPart}(P)$ and $\texttt{vPart}(P)$ the *corner* of P and denote it by $\texttt{corner}(P)$. Moreover, let $\texttt{hTip}(P)$ (resp., $\texttt{vTip}(P)$) denote the endpoint of $\texttt{hPart}(P)$ (resp., $\texttt{vPart}(P)$) that is not shared with $\texttt{vPart}(P)$ (resp., not shared with $\texttt{hPart}(P)$); see the figure on this page for an example. Let $N[P]$ denote the set of paths adjacent to P; we assume that

$P \in N[P]$. Moreover, for a set S of paths, define $N[S] := \cup_{P \in S} N[P]$. We denote the set of neighbours of P that share at least one grid edge (or, a grid node) with $\texttt{hPart}(P)$ (resp., with $\texttt{vPart}(P)$) by $\texttt{hNeighbor}(P)$ (resp., by $\texttt{vNeighbor}(P)$).

3 Domination on B_1-VPG Graphs

Recall that the Minimum Dominating Set problem is known to be APX-hard on one-string B_1-VPG graphs. In the following, we give the first $O(1)$-approximation algorithm for Minimum Dominating Set on one-string B_1-VPG graphs. For the rest of this section, let G be a one-string B_1-VPG graph with n vertices, and let $\langle \mathcal{P}_{\texttt{vtx}}, \mathcal{G} \rangle$ denote its string representation.

Our algorithm is based on first formulating the problem on G as a hitting set problem and then computing a small ε-net for the corresponding instance of the hitting set problem. Having such an ε-net along with the technique of Brönnimann and Goodrich [4] gives us an $O(1)$-approximation algorithm for Minimum Dominating Set on G.

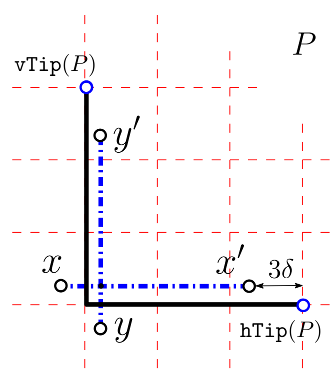
A *set system* is a pair $\mathcal{R} = (\mathcal{U}, \mathcal{S})$, where $\mathcal{U}$ is a ground set of elements and $\mathcal{S}$ is a collection of subsets of $\mathcal{U}$. A *hitting set* for the set system $(\mathcal{U}, \mathcal{S})$ is a subset M of $\mathcal{U}$ such that $M \cap S \neq \emptyset$ for all $S \in \mathcal{S}$; we call each element of $\mathcal{U}$ a *hitting element*.

For the Minimum Dominating Set problem on G, we construct the set system $(\mathcal{U}, \mathcal{S})$ as follows. Let P be a ⌞-type path. We associate a *cross* $c := (\ell_H, \ell_V)$ with P in which ℓ_H and ℓ_V denote its horizontal and vertical segments, respectively; we call ℓ_H and ℓ_V the *supporting segments* of c. The left endpoint of ℓ_H is

$$x := (x(\texttt{corner}(P)) - \delta, y(\texttt{corner}(P)))$$

and its right endpoint is

$$x' := ((x(\texttt{hTip}(P)) - 3\delta, y(\texttt{corner}(P)))$$



in which $\delta := w/4$, where w is the length of a grid edge. See the figure on the right. Analogously, the bottom endpoint of ℓ_V is $y := (x(\texttt{corner}(P), y(\texttt{corner}(P)) - \delta)$ and its top endpoint is $y' := ((x(\texttt{corner}(P)), y(\texttt{vTip}(P)) - 3\delta)$. The cross of an $\texttt{x}$-type path, where $\texttt{x} \in$ {⌜, ⌝, ⌟}, is defined analogously. We denote the cross of a path P by $\texttt{cross}(P)$. The following observation is immediate by the construction of a cross.

Observation 1. *Let P_u, P_v be two paths in $\mathcal{P}_{\texttt{vtx}}$. Then, P_u and P_v intersect each other at a grid node if and only if $\texttt{cross}(P_u)$ and $\texttt{cross}(P_v)$ intersect each other.*

To see the hitting elements of the set system, for each path $P \in \mathcal{P}_{\texttt{vtx}}$, we add into $\mathcal{U}$ both of the supporting segments of $\texttt{cross}(P)$. We say that a hitting element $e \in \mathcal{U}$ *hits* a cross c if e intersects one of the supporting segments of c; we assume that the hitting elements corresponding to supporting segments of c also hit c. To compute $\mathcal{S}$, consider the set of elements of $\mathcal{U}$ hitting a cross c and let $\mathcal{S}$ be the collection of these sets; notice that there is a set in $\mathcal{S}$ for each cross c. This forms the set system $(\mathcal{U}, \mathcal{S})$ corresponding to Minimum Dominating Set on G.

Lemma 1. *If there is a dominating set of size k on the one-string* B_1-VPG *graph G, then there is a hitting set of size at most $2k$ on the set system $(\mathcal{U}, \mathcal{S})$. Moreover, if there is a hitting set of size k on the set system $(\mathcal{U}, \mathcal{S})$, then there is a dominating set of size at most k on G.*

Proof. Let M be a dominating set of size k for G. For each path $P \in M$, add the supporting segments of $\texttt{cross}(P)$ into M'. Clearly, $|M'| = 2k$. Now, consider a set $S \in \mathcal{S}$ and let P and c denote its corresponding path and cross, respectively. Since M is a dominating set, either $P \in M$ or $P' \in M$ for some path P' intersecting P. If $P \in M$, then clearly S is intersected by some segment in M'. If $P' \in M$, then c intersects $\texttt{cross}(P')$ by Observation 1 and so at least one of the supporting segments of $\texttt{cross}(P')$ (that is in M') intersects S.

Now, suppose that M' is a hitting set of size k for the set system $(\mathcal{U}, \mathcal{S})$. For each hitting element $e \in M'$: add P to M, where P is the path for which e is a supporting segment of $\texttt{cross}(P)$. Clearly, $|M| \leq k$. To see why M is a dominating set for G, consider any path P and let $S \in \mathcal{S}$ denote its corresponding set. Since M' is a hitting set, there must be a hitting element $e' \in M'$ that intersects S. Let P' be the path for which e' is a supporting segment of $\texttt{corner}(P')$; notice that $P' \in M$. If $P' = P$, then we are done. Otherwise, e' must hit the cross c corresponding to P since $c \in S$. This means that the cross corresponding to P' intersects c and so by Observation 1 P is intersected by P'. $\square$

An *ε-net* for a set system $\mathcal{R} = (\mathcal{U}, \mathcal{S})$ is a subset N of $\mathcal{U}$ such that every set S in $\mathcal{S}$ with size at least $\varepsilon \cdot |\mathcal{U}|$ has a non-empty intersection with N. Brönnimann and Goodrich [4] introduced an iterative-doubling approach to turn algorithms for finding ε-nets into approximation algorithms for hitting sets of minimum size (see the full version of the paper for a description of this approach).

Theorem 1. *[4] Let $\mathcal{R}$ be a set system that admits both a polynomial-time net finder and a polynomial-time verifier. Then, there exists a polynomial-time algorithm that computes a hitting set of size at most $s(4 \cdot \mathrm{OPT})$, where* OPT *is the size of a minimum hitting set, and $s(r)$ is the size of the $(1/r)$-net found by the net finder.*

Clearly, the hitting set problem corresponding to Minimum Dominating Set on G has a polynomial-time verifier. In the following, we compute an ε-net of size $O(1/\varepsilon)$ for the set system $(\mathcal{U}, \mathcal{S})$, which combined by Lemma 1 and Theorem 1 gives an $O(1)$-approximation algorithm for Minimum Dominating

Set on G. To compute the ε-net, we first compute such an ε-net for a "one-dimensional" variant of our hitting set problem and then show that re-using such an ε-net twice would result in the desired ε-net. Recall the set system $(\mathcal{U}, \mathcal{S})$ corresponding to Minimum Dominating Set on G that we constructed before Lemma 1. Let $\mathcal{U}_H$ denote the set of only-horizontal hitting elements of $\mathcal{U}$, and consider the set system $(\mathcal{U}_H, \mathcal{S}_H)$ in which $\mathcal{S}_H$ is defined analogous to $\mathcal{S}$. Representing each cross by its vertical supporting segment only, the minimum hitting set problem on $(\mathcal{U}_H, \mathcal{S}_H)$ reduces to the following problem: given a set of horizontal line segments $\mathcal{H}$ and a set of vertical line segments $\mathcal{V}$, find a minimum-cardinality set $S \subseteq \mathcal{H}$ such that every line segment in $\mathcal{V}$ is intersected by S. This problem is known as the *orthogonal segment covering* problem and is shown to be NP-complete [13].

Lemma 2. *The minimum hitting set problem on $(\mathcal{U}_H, \mathcal{S})$ reduces to the orthogonal segment covering problem.*

Proof. ($\Rightarrow$) Let M be a feasible solution for the minimum hitting set problem on $(\mathcal{U}_H, \mathcal{S})$. If a hitting element e hits a cross c, then it intersects one of its supporting segments s. Let P denote the path in $\mathcal{P}_{\texttt{vtx}}$ corresponding to cross c. Since e is horizontal, s cannot be horizontal because if e intersects the horizontal supporting segment of P (i.e., s), then $\texttt{vPart}(P)$ and $\texttt{hPart}(P')$ must intersect each other in more than one point, where P' is the path in $\mathcal{P}_{\texttt{vtx}}$ for which e is a supporting segment. This contradicts the fact that G is a one-string B_1-EPG graph. Therefore, s is vertical and so for any feasible solution M, we have a feasible solution for the orthogonal segment covering problem with the same size.

($\Leftarrow$) Clearly, a feasible solution to the minimum segment covering problem is also a feasible solution to the minimum hitting set problem with the same size. □

For the orthogonal segment covering problem, Biedl et al. [3] showed that there exists an ε-net of size $s(1/\varepsilon) \in O(1/\varepsilon)$. Therefore, by Lemma 2, we have the following result.

Lemma 3. *There exists an ε-net of size $O(1/\varepsilon)$ for the minimum hitting set problem on $(\mathcal{U}_H, \mathcal{S}_H)$.*

Lemma 4. *There exists a polynomial-time $O(1)$-approximation algorithm for the minimum hitting set problem on $(\mathcal{U}, \mathcal{S})$.*

Proof. To prove the lemma, it suffices by Theorem 1 to compute an ε-net of size $O(1/\varepsilon)$ for $(\mathcal{U}, \mathcal{S})$. Define the set system $(\mathcal{U}_V, \mathcal{S}_V)$ similar to $(\mathcal{U}_H, \mathcal{S}_\mathcal{H})$ (i.e., $\mathcal{U}_V$ is the set of only-vertical hitting elements of $\mathcal{U}$ and define $\mathcal{S}_V$ analogously). By Lemma 3, let N_H (resp., N_V) be an $(\varepsilon/2)$-net of size $O(1/\varepsilon)$ for the corresponding one-dimensional variant of the hitting set problem on $(\mathcal{U}_H, \mathcal{S}_H)$ (resp., $(\mathcal{U}_V, \mathcal{S}_V)$), and let $N := N_H \cup N_V$. Clearly, N has size $O(1/\varepsilon)$. We now prove that N is an ε-net.

Let $S \in \mathcal{S}$ such that $|S| \geq \varepsilon \cdot |\mathcal{U}|$. We need to show that $N \cap S \neq \emptyset$. Notice that having $|S| \geq \varepsilon \cdot |\mathcal{U}|$ means that there exists a cross c that is hit by $\varepsilon \cdot |\mathcal{U}|$ hitting elements. Assume w.l.o.g. that at least half of these hitting elements are horizontal. Then, the vertical supporting segment of c intersects at least $\varepsilon \cdot |\mathcal{U}/2|$ horizontal hitting elements of $\mathcal{U}$. By definition of an $(\varepsilon/2)$-net, there is a hitting element $e \in N_H$ that intersects the vertical supporting segment of c. Therefore, e hits cross c and so $e \in N \cap S$. □

Putting everything together, we can prove the main result of this section.

Theorem 2. *There exists a polynomial-time $O(1)$-approximation algorithm for* Minimum Dominating Set *on any one-string* B_1-VPG *graph.*

Proof. Let G be any one-string B_1-VPG graph with its string representation $\langle \mathcal{P}_{\mathtt{vtx}}, \mathcal{G} \rangle$. To prove the theorem, it is sufficient to show that there exists a polynomial-time algorithm that finds a constant-factor approximation of Minimum Dominating Set in G.

First, compute the set system $(\mathcal{U}, \mathcal{S})$ corresponding to $\mathcal{P}_{\mathtt{vtx}}$ as described above. Let OPT_{DS} (resp., OPT_{HS}) denote an optimal solution for Minimum Dominating Set on G (resp., for the minimum hitting set problem on $(\mathcal{U}, \mathcal{S})$). By Lemma 1, we know that $|\mathrm{OPT}_{HS}| \leq 2|\mathrm{OPT}_{DS}|$. By Lemma 4, let S_{HS} be a constant-factor approximation to the minimum hitting set problem on $(\mathcal{U}, \mathcal{S})$; that is, $|S_{HS}| \leq d \cdot |\mathrm{OPT}_{HS}|$ for some constant $d > 0$. Apply Lemma 1 to S_{HS} and let S_{DS} be a feasible solution for Minimum Dominating Set on G; notice that $|S_{DS}| \leq |S_{HS}|$. The set system $(\mathcal{U}, \mathcal{S})$, S_{HS} and S_{DS} can each be computed in polynomial time. Therefore,

$$\frac{|S_{DS}|}{|\mathrm{OPT}_{DS}|} \leq \frac{|S_{HS}|}{|\mathrm{OPT}_{DS}|} \leq \frac{2|S_{HS}|}{|\mathrm{OPT}_{HS}|} \leq 2d.$$

That is, S_{DS} is an $O(1)$-approximation of Minimum Dominating Set on G, which can be computed in polynomial time. □

4 Domination on B_1-EPG Graphs

In this section, we consider the Minimum Dominating Set problem on B_1-EPG graphs. We first show that the problem is APX-hard, even if only two types of paths are allowed in the graph; hence, ruling out the possibility of a PTAS for this problem unless P=NP. Recall that a 4-approximation algorithm is already known for Minimum Dominating Set on B_1-EPG graphs [5, 12]. Here, in Section 4.2, we give c-approximation algorithms for this problem on two subclasses of B_1-EPG graphs, for small values of c.

4.1 APX-Hardness

To show the APX-hardness, we give an L-reduction [16] from the Minimum Vertex Cover problem on graphs with maximum-degree three to the Minimum Dominating Set on B_1-EPG graphs; Minimum Vertex Cover is known to be APX-hard on graphs with maximum-degree three [1].

Lemma 5. Minimum Vertex Cover *on graphs with maximum-degree three is* L*-reducible to* Minimum Dominating Set *on* B_1-EPG *graphs.*

Proof. Consider an arbitrary instance I of Minimum Vertex Cover on graphs of maximum-degree three; let $G = (V, E)$ be the graph corresponding to I and let k be the size of the smallest vertex cover in G. First, let $u_1, \ldots, u_n$ be an arbitrary ordering of the vertices of G, where $n = |V|$. In the following, we give a computable function f that takes I as input and outputs an instance $f(I)$ of Minimum Dominating Set in polynomial time, where $f(I)$ consists of a B_1-EPG graph such that its paths are of either ⌜-type or ⌟-type only.

We first describe the vertex gadgets. For each vertex u_i, $1 \le i \le n$, construct a horizontal ⌜-type Γ_i^h and a vertical ⌜-type Γ_i^v, and connect them as shown in Figure 2. We call the big ⌜-type path used in the connection of Γ_i^h and Γ_i^v the *big connector* C_i of i, and the two small (blue, dashed) ⌜-type paths the *small connectors* of i. For each edge $(u_i, u_j) \in E$, where $i < j$, we add two small paths, one of ⌜-type and one of ⌟-type, at the intersection point of Γ_i^v and Γ_j^h such that each of them becomes adjacent to both Γ_i^v and Γ_j^h; see the two (red, dash-dotted) ⌜-type and ⌟-type paths at the intersection of Γ_1^v and Γ_2^h in Figure 2. We denote this pair of paths by $E_{i,j}$; notice that the paths of $E_{i,j}$ are not adjacent to each other (they only share a grid node). This gives the instance $f(I)$ of Minimum Dominating Set on B_1-EPG graphs; let G' be the corresponding B_1-EPG graph. Notice that f is a polynomial-time computable function. In the following, we denote an optimal solution for the instance X of a problem by $s^*(X)$. We now prove that all the four conditions of L-reduction hold.

First, let M be a vertex cover of G of size k. Denote by $\Gamma^h[M] := \{\Gamma_i^h | u_i \in M\}$ the set of horizontal paths induced by M and define $\Gamma^v[M]$ analogously. Moreover, let $C[M] := \{C_i | u_i \notin M\}$ be the set of big connectors whose corresponding vertex is not in M. We show that $D := \Gamma^h[M] \cup \Gamma^v[M] \cup C[M]$ is a dominating set of G'. Let γ be a path. If γ is any of the paths in $E_{i,j}$ for some i, j, then $u_i \in M$ or $u_j \in M$ because M is a vertex cover; assume w.l.o.g. that $u_i \in M$. Then, $\Gamma_i^h, \Gamma_i^v \in D$ and so γ must be dominated. If γ is a connector of i for some i (either big or small), then there are two cases: if $u_i \in M$, then $\Gamma_i^h, \Gamma_i^v \in D$ and so the connector is dominated. If $u_i \notin M$, then $C_i \in C[M]$ and $C[M] \subseteq D$; hence, the connector is again dominated. Finally, suppose that γ is Γ_i^h (resp., Γ_i^v) for some i. If $u_i \in M$, then $\Gamma_i^h, \Gamma_i^v \in D$ and so Γ_i^h (resp., Γ_i^v) is dominated. If $u_i \notin M$, then $C_i \in C[M]$ and $C[M] \subseteq D$; hence, Γ_i^h (resp., Γ_i^v) is again dominated. This shows that D is a dominating set for G'.

Second, let D be an arbitrary dominating set on G'. First, notice that we can construct a dominating set D' for G' such that $|D'| \le |D|$ and D' consists of only Γ_i^h and Γ_i^v for some i, or a big connector. This is because (i) any path dominated by a small connector is also dominated by some big connector, and (ii) any path dominated by a path from $E_{i,j}$ is also dominated by Γ_i^v or Γ_j^h. For (ii), in particular, if exactly one of the paths in $E_{i,j}$ is in D, then at least one of Γ_i^v or Γ_j^h must be in D in order to dominate the other path of $E_{i,j}$; hence, we can replace the path of $E_{i,j}$ in D with one of Γ_i^v or Γ_j^h arbitrarily.

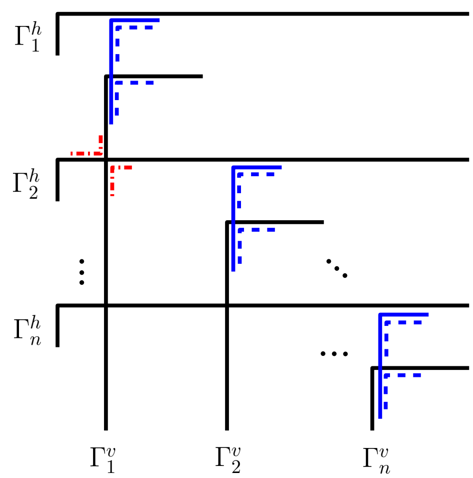


Fig. 2: An illustration in support of the construction in the proof of Lemma 5.

Otherwise, if both paths of $E_{i,j}$ are in D, then we can replace both of them with Γ_i^v and Γ_j^h. So, $|D'| \leq |D|$ and D' is a feasible dominating set for G'. Now, define $\Gamma_{\texttt{both}}[D'] = \{\Gamma_i^h, \Gamma_i^v | \Gamma_i^h, \Gamma_i^v \in D'\}$; i.e., the paths of a vertex u_i, where *both* its horizontal and its vertical copies appear in D'. Also, define $\Gamma_{\texttt{one}}[D']$ to be the remaining paths of type Γ_i^h and Γ_i^v; i.e., those of u_i, where *only* one of its copies appears in D'. Finally, let $C[D']$ be the set of big connectors in D'. We denote $\Gamma_{\texttt{both}}[D'] \cup \Gamma_{\texttt{one}}[D']$ by $\Gamma[D']$. Now, let $M := \{u_i | \Gamma_i^h \in D' \text{ or } \Gamma_i^v \in D'\}$. Since all $E_{i,j}$ are dominated by $\Gamma[D']$, M is a vertex cover.

Third, observe that $|\Gamma^h[M]| = |\Gamma^v[M]| = |M| = k$ and also $|C[M]| = n - k$. Given that G has degree three, $k \geq n/4$ and so $|s^*(f(I))| \leq n - k + k + k \leq 5k \leq 5|s^*(I)|$.

Finally, to dominate all connectors of i, we must have $C_i \in D'$ or $\Gamma_i^h, \Gamma_i^v \in D'$; this indeed holds for all i. Thus, $|C[D']| + |\Gamma_{\texttt{both}}[D']|/2 \geq n$. Moreover, $|\Gamma_{\texttt{one}}[D']| + |\Gamma_{\texttt{both}}[D']|/2 \geq k$ since M is a vertex cover of G. Therefore, $|D'| \geq |\Gamma_{\texttt{both}}[D']| + |\Gamma_{\texttt{one}}[D']| + |C[D']| \geq |\Gamma_{\texttt{one}}[D']| + |\Gamma_{\texttt{both}}[D']|/2 + n \geq k + n$. By this and our earlier inequality $|s^*(f(I))| \leq n - k + k + k$, we have $|s^*(f(I))| = n + k$. Now, suppose that $|D| = |s^*(f(I))| + c$ for some $c \geq 0$. Then,

$$
\begin{aligned}
&|D| - |s^*(f(I))| = c \\
&\Rightarrow |D| - (n + k) = c \\
&\Rightarrow |D'| - (n + k) \leq c \\
&\Rightarrow |\Gamma_{\texttt{one}}[D']| + |\Gamma_{\texttt{both}}[D']|/2 + n - (n + k) \leq c \\
&\Rightarrow |\Gamma_{\texttt{one}}[D']| + |\Gamma_{\texttt{both}}[D']|/2 - k \leq c \\
&\Rightarrow |M| - |s^*(I)| \leq c.
\end{aligned}
$$

That is, $|M| - |s^*(I)| \leq |D| - |s^*(f(I))|$. This concludes our L-reduction from Minimum Vertex Cover on graphs of maximum-degree three to Minimum Dominating Set on B_1-EPG graphs with $\alpha = 5$ and $\beta = 1$. □

Our reduction reveals that every path in the constructed B_1-EPG graph G' is a ⌜-type or a ⌟-type path. Therefore, by Lemma 5, we have the following theorem.

Theorem 3. *The* Minimum Dominating Set *problem is* APX*-hard on* B_1-EPG *graphs, even if all the paths in the graph are of type* ⌜ *or* ⌟*. Thus, there is no PTAS for this problem on* B_1-EPG *graphs unless* P=NP*.*

4.2 Approximation Algorithms

In this section, we give constant-factor approximation algorithms for the Minimum Dominating Set problem on two subclasses of B_1-EPG graphs. Let us first define these subclasses.

First, we consider a subclass of B_1-EPG graphs in which every path of each B_1-EPG graph intersects two axis-parallel lines that are normal to each other. We notice that this variant has already been considered, where Lahiri et al. [14] gave an exact solution for the Maximum Independent Set problem when the input graph is a B_1-VPG graph: they showed that the induced graph is a co-comparability graph and so solved the Maximum Independent Set problem exactly. However, the graph induced by this variant when considering B_1-EPG graphs is not necessarily a co-comparability graph; this is mainly because two paths intersecting in only one point in a B_1-EPG graph are not adjacent. Here, we give a 2-approximation algorithm for this problem on such B_1-EPG graphs. We call this subclass, the class of Double-Crossing B_1-EPG graphs. Next, we consider a less-restricted subclass of B_1-EPG graphs in which every path of each B_1-EPG graph intersects only a vertical line. We show that the same algorithm is a 3-approximation algorithm for the problem on this subclass of B_1-EPG graphs, albeit considering a "non-containment" assumption. We call this subclass, the class of Vertical-Crossing B_1-EPG graphs.

Before describing the algorithms, let us define an ordering $\prec$ on the paths in $\mathcal{P}_{\texttt{edg}}$ as follows. The paths appear in the ordering by the y-coordinate of their corners from bottom to top and then from left to right whenever they have the same y-coordinate; that is, $P \prec P'$ for two paths in $\mathcal{P}_{\texttt{edg}}$, if and only if $y(\texttt{corner}(P)) < y(\texttt{corner}(P'))$ or $y(\texttt{corner}(P)) = y(\texttt{corner}(P'))$ but $x(\texttt{corner}(P)) < x(\texttt{corner}(P'))$; we break ties arbitrarily to complete the ordering. For the rest of this section, we assume that every path in $\mathcal{P}_{\texttt{edg}}$ is a ⌞-type path.

Double-crossing B_1-EPG *graphs.* For a Double-Crossing B_1-EPG graph, we are given a B_1-EPG graph G, a horizontal line L_1 and a vertical line L_2 both on the grid $\mathcal{G}$ such that L_1 and L_2 intersect each other and P intersects both L_1 and L_2 for all $P \in \mathcal{P}_{\texttt{edg}}$ (hence, $\texttt{corner}(P)$ lies in the lower-left quadrant defined

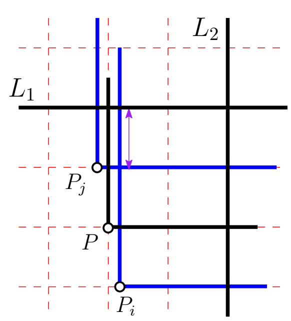
Algorithm 1 ApproximateLineMDS(G, L_1, L_2)

$S \leftarrow \emptyset$;
for each path $P \in \mathcal{P}_{\texttt{edg}}$ in increasing order $\prec$ **do**
  $S \leftarrow S \cup P$;
  $\mathcal{P}_{\texttt{edg}} \leftarrow \mathcal{P}_{\texttt{edg}} \setminus N[P]$;
return S;

by L_1 and L_2). Our 2-approximation algorithm for the Minimum Dominating Set problem is as follows; let S be an initially-empty set. For each path P in the increasing order $\prec$: add P into S and set $\mathcal{P}_{\texttt{edg}} := \mathcal{P}_{\texttt{edg}} \setminus N[P]$. See Algorithm 1. Clearly, the algorithm terminates in time polynomial in $|\mathcal{P}_{\texttt{edg}}|$, and S is a feasible solution for the problem. To see the approximation factor, let OPT be an optimal solution for the Minimum Dominating Set problem on G; notice that by deleting the paths in $S \cap OPT$ we can assume that $S \cap OPT = \emptyset$. This means that every path in S must be adjacent to at least one path in OPT.

Lemma 6. *Any path in OPT is adjacent to at most two distinct paths in S.*

Proof. Suppose for a contradiction that there exists a path $P \in OPT$ that is adjacent to three distinct paths P_1, P_2 and P_3 of S, where $P_1 \prec P_2 \prec P_3$. W.l.o.g., assume that $P_i, P_j \in \texttt{vNeighbor}(P)$ for some $i < j \in \{1, 2, 3\}$. This means that $x(\texttt{corner}(P)) = x(\texttt{corner}(P_i)) = x(\texttt{corner}(P_j))$. Since $i < j$, we have that $y(\texttt{corner}(P_i)) < y(\texttt{corner}(P_j))$. Moreover, since all the paths in $\mathcal{P}_{\texttt{edg}}$ intersect the horizontal line L_1, the three paths P, P_i and P_j must all intersect L_1 at the same point and so they share the top-most vertical grid edge below L_1 on which $\texttt{vPart}(P)$ lies; see the figure on the right. Thus, P_i and P_j are adjacent in G. Since $y(\texttt{corner}(P_i)) < y(\texttt{corner}(P_j))$, we have $P_i \prec P_j$ and so P_j is removed from $\mathcal{P}_{\texttt{edg}}$ when the algorithm adds P_i to S. So, $P_j \notin S$ — a contradiction. ☐



Since every path in S must be adjacent to at least one path in OPT and any path in OPT can be adjacent to at most two distinct paths in S by Lemma 6, we have $|S| \leq 2|OPT|$. Therefore, we have the following theorem.

Theorem 4. *There exists a polynomial-time 2-approximation algorithm for the* Minimum Dominating Set *problem on* Double-Crossing B_1-EPG *graphs.*

Vertical-crossing B_1-EPG *graphs.* Here, we are given a B_1-EPG graph G and a vertical line ℓ on the grid $\mathcal{G}$ such that $\texttt{hPart}(P)$ intersects ℓ for all $P \in \mathcal{P}_{\texttt{edg}}$. Moreover, we make a *non-containment assumption* in the sense that the vertical segment of no path is entirely contained in that of any other path in $\mathcal{P}_{\texttt{edg}}$; that is, for every two paths $P, P' \in \mathcal{P}_{\texttt{edg}}$ such that $P \in \texttt{vNeighbor}(P')$, neither

$\texttt{vPart}(P) \subseteq \texttt{vPart}(P')$ nor $\texttt{vPart}(P') \subseteq \texttt{vPart}(P)$. We prove that Algorithm 1 is a 3-approximation algorithm for the MINIMUM DOMINATING SET problem on VERTICAL-CROSSING B_1-EPG graphs.

Theorem 5. *Algorithm 1 is a 3-approximation algorithm for the* MINIMUM DOMINATING SET *problem on* VERTICAL-CROSSING B_1-EPG *graphs under the non-containment assumption.*

Proof. Let G be any VERTICAL-CROSSING B_1-EPG graph. Moreover, let OPT be an optimal solution for the MINIMUM DOMINATING SET problem on G and let S be the solution returned by Algorithm 1. Again, we can assume that $S \cap OPT = \emptyset$. Thus, every path in S must be dominated by at least one path in OPT. In the following, we show that any path in OPT can dominate at most three distinct paths in S and so prove that $|S| \leq 3|OPT|$. Let P be any path in OPT. We show that $|S \cap \texttt{hNeighbor}(P)| \leq 1$ and $|S \cap \texttt{vNeighbor}(P)| \leq 2$.

First, suppose for a contradiction that there are two paths $P_1, P_2 \in S \cap \texttt{hNeighbor}(P)$; assume w.l.o.g. that $P_1 \prec P_2$. Notice that $y(\texttt{corner}(P)) = y(\texttt{corner}(P_1)) = y(\texttt{corner}(P_2))$. Thus, since all the three paths P, P_1 and P_2 intersect the vertical line ℓ, we conclude that they all intersect ℓ at the same point. Therefore, they share the rightmost horizontal grid edge to the left of ℓ on which $\texttt{hPart}(P)$ lies. This means that P_1 and P_2 are adjacent in G. Since $P_1 \prec P_2$, Algorithm 1 removes P_2 from $\mathcal{P}_{\texttt{edg}}$ when adding P_1 into S; that is, $P_2 \notin S$ — a contradiction. So, $|S \cap \texttt{hNeighbor}(P)| \leq 1$.

Now, suppose for a contradiction that there are three paths $P_1, P_2, P_3 \in S \cap \texttt{vNeighbor}(P)$; assume w.l.o.g. that $P_1 \prec P_2 \prec P_3$. Notice that $x(\texttt{corner}(P)) = x(\texttt{corner}(P_i))$ for all $1 \leq i \leq 3$. Consider $y(\texttt{corner}(P))$. Then, for at least two paths P_i, P_j, where $i < j \in \{1, 2, 3\}$, we have

$$y(\texttt{corner}(P_i)), y(\texttt{corner}(P_j)) < y(\texttt{corner}(P))$$

or

$$y(\texttt{corner}(P_i)), y(\texttt{corner}(P_j)) \geq y(\texttt{corner}(P)).$$

- If $y(\texttt{corner}(P_i)), y(\texttt{corner}(P_j)) < y(\texttt{corner}(P))$, then we know $\texttt{vPart}(P_i)$ and $\texttt{vPart}(P_j)$ both share with P the bottom-most vertical grid edge on which $\texttt{vPart}(P)$ lies, implying that P_i and P_j are adjacent in G. Since $i < j$, we have $P_i \prec P_j$ and so Algorithm 1 removes P_j from G when adding P_i to S. So, $P_j \notin S$ — a contradiction.
- If $y(\texttt{corner}(P_i)), y(\texttt{corner}(P_j)) \geq y(\texttt{corner}(P))$, then

$$y(\texttt{vTip}(P_i)) > y(\texttt{vTip}(P)) \text{ and } y(\texttt{vTip}(P_j)) > y(\texttt{vTip}(P)),$$

  because otherwise $\texttt{vPart}(P_i) \subseteq \texttt{vPart}(P)$ or $\texttt{vPart}(P_j) \subseteq \texttt{vPart}(P)$, which is a contradiction to the non-containment assumption of paths in G. Since $i < j$, we have $P_i \prec P_j$. Therefore, P_i and P_j share the top-most vertical grid edge on which $\texttt{vPart}(P)$ lies, meaning that P_i and P_j are adjacent in G. So, Algorithm 1 removes P_j when adding P_i into S — a contradiction to $P_j \in S$.

Therefore, $|S \cap \texttt{hNeighbor}(P)| \leq 1$ and $|S \cap \texttt{vNeighbor}(P)| \leq 2$. This completes the proof of the theorem. □

5 Conclusion

In this paper, we studied the Minimum Dominating Set problem on B_1-VPG and B_1-EPG graphs. For B_1-VPG graphs, we gave an $O(1)$-approximation algorithm for this problem when the input graph is one-string. For B_1-EPG graphs, we proved that Minimum Dominating Set is APX-hard (even if the graph has only two types of paths), ruling out the existence of a PTAS unless P=NP. We also gave c-approximation algorithms for this problem on two subclasses of B_1-EPG graphs, for $c \in \{2, 3\}$. We conclude the paper by the following open problems:

- Our $O(1)$-approximation algorithm for Minimum Dominating Set on one-string B_1-VPG graphs relies on the fact that the input graph is one-string (in the proof of Lemma 2, in particular); is there an $O(1)$-approximation algorithm for this problem on any B_1-VPG graph?
- Is the Minimum Dominating Set problem APX-hard on B_1-EPG graphs, if the graph consists of only one type of paths? We believe that a slight modification to our APX-hardness result would answer this question affirmatively.
- Our 2- and 3-approximation algorithms for Minimum Dominating Set work only on the described subclasses of B_1-EPG graphs; is there an α-approximation algorithm for the Minimum Dominating Set problem on any B_1-EPG graph, for some $\alpha < 4$? (Recall that a 4-approximation algorithm is already known [5, 12].)
- Is the Minimum Dominating Set problem NP-hard on Vertical-Crossing B_1-EPG graphs?

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