

1-Bend RAC Drawings of 1-Planar Graphs [★]

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Abstract. A graph is *1-planar* if it has a drawing where each edge is crossed at most once. A drawing is *RAC* (*Right Angle Crossing*) if the edges cross only at right angles. The relationships between 1-planar graphs and RAC drawings have been partially studied in the literature. It is known that there are both 1-planar graphs that are not straight-line RAC drawable and graphs that have a straight-line RAC drawing but that are not 1-planar [22]. Also, straight-line RAC drawings always exist for *IC-planar graphs* [9], a subclass of 1-planar graphs. One of the main questions still open is whether every 1-planar graph has a RAC drawing with at most one bend per edge. We positively answer this question.

1 Introduction

An emerging research line in Graph Drawing studies families of non-planar graphs that can be drawn so that crossing edges verify some desired properties. This topic is informally recognized as “beyond planarity”. Different types of properties give rise to different families of beyond planar graphs. Among them, particular attention has been devoted to *1-planar graphs* (see, e.g., [1, 2, 7–9, 17, 23, 24, 29, 31, 35]) and to *RAC* (*Right Angle Crossing*) *graphs* (see, e.g., [4, 6, 13–15, 18–21, 30, 27]). A graph is 1-planar if it has a drawing where each edge is crossed at most once, while it is RAC if it has a polyline drawing where the edges cross only at right angles. From an application point of view, the study of these two families is motivated by several cognitive experiments, suggesting that the readability of a layout is negatively correlated to the number of crossings [33, 34, 38] and that user task performances are not affected too much if edges cross at large angles [25, 26, 28]. Also, users often prefer straight-line drawings or layouts whose edges have few bends [32], and several algorithms optimize this aesthetic criterion [11]. Note that, every graph admits a polyline RAC drawing with at most three bends per edge [18].

For the reasons above, it is interesting to study what graphs can be drawn with at most one crossing per edge, right angle crossings, and few bends per

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edge at the same time. We recall that n -vertex 1-planar graphs have at most $4n - 8$ edges [31] and that straight-line 1-planar drawings have at most $4n - 9$ edges [17]. Also, straight-line RAC graphs have at most $4n - 10$ edges [18], while RAC drawings with at most one bend per edge or two bends per edge, have at most $6.5n - 13$ and $74.2n$ edges, respectively [5]. These results immediately imply that there are 1-planar graphs not admitting 1-planar drawings with straight-line edges and 1-planar graphs not admitting straight-line drawings with right angle crossings. Also, there exist straight-line RAC drawable graphs that are not 1-planar [22]. In this scenario, one of the main questions still open is whether every 1-plane graph admits a RAC drawing with at most one bend per edge. This paper positively answers this question, by proving the following result.

Theorem 1. *Let G be an n -vertex 1-planar graph. Then G admits a 1-planar RAC drawing Γ with at most one bend per edge. Also, if a 1-planar embedding of G is given as part of the input, Γ can be computed in $O(n)$ time.*

We remark that a characterization of the 1-planar graphs that can be drawn with straight-line edges was given by Thomassen in 1988 [37]. The characterization is described in terms of the existence of a 1-planar embedding that does not contain two primitive forbidden configurations. This result immediately implies that every 1-planar graph admits a 1-planar drawing with at most one bend per edge (which is not necessarily RAC); it is sufficient to subdivide each crossing edge of any given 1-planar embedding with a dummy vertex, so to remove any possible forbidden configuration. Dummy vertices will correspond to bends in the final drawing. Moreover, Alam et al. [2], proved that every 3-connected 1-plane graph can be drawn with straight-line edges, except for at most one edge that may require one bend. We also remark that straight-line RAC drawings always exist for *IC-planar graphs* [9], a subclass of 1-planar graphs.

Some proofs and technicalities are omitted and can be found in [16].

2 Preliminaries

We assume familiarity with basic terminology of graph drawing [11]. In the following we only consider *simple* drawings of graphs, i.e., drawings where two edges have at most one point in common (which is either a common endpoint or a common interior point where the two edges properly cross each other). A *k-bend drawing* of a graph is a drawing where each edge is represented as a polyline with at most $k > 0$ bends. A graph G is *planar* if it admits a planar (i.e., crossing-free) drawing. Such a drawing subdivides the plane into topologically connected regions, called *faces*. The infinite region is the *outer face*. The number of vertices encountered in the closed walk along the boundary of a face f is the *degree* of f . If G is not 2-connected a vertex may be encountered more than once, thus contributing with more than one unit to the degree of f . A *planar embedding* of G is an equivalence class of planar drawings of G having the same set of faces. A *plane graph* is a planar graph with a given planar embedding.

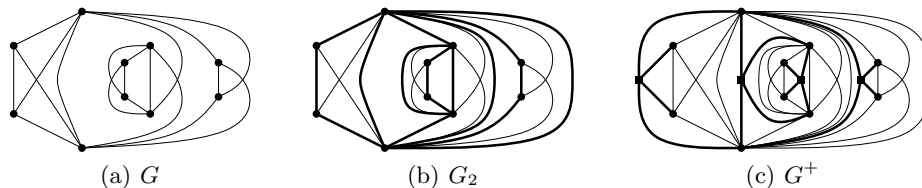


Fig. 1. Illustration for the augmentation step.

The concept of planar embedding can be extended to non-planar drawings. Given a non-planar drawing Γ , interpret every crossing as a vertex. The resulting planarized drawing has a planar embedding. An *embedding* of a (non-planar) graph G is an equivalence class of drawings whose planarized versions have the same planar embedding. A *1-plane* graph is a 1-planar graph with a given *1-planar embedding*, i.e., an embedding where each edge is crossed at most once. Each face of a 1-plane embedding is composed of both vertices and/or crossings, and its degree is the number of vertices or crossings encountered in the closed walk along its boundary. A *kite* K is a 1-plane graph isomorphic to K_4 with an embedding such that all the vertices are on the boundary of the outer face, the four edges on the boundary are crossing-free, and the remaining two edges cross each other. Given a 1-plane graph G and a kite $K = \{a, b, c, d\}$, such that $K \subseteq G$, we say that K is *empty* if it does not contain any vertex of G inside the 4-cycle $\{a, b, c, d\}$ (it contains only the crossing point). A pair of crossing edges of G forms an *empty kite* if their four end-vertices induce an empty kite. A 1-plane graph G , possibly containing parallel edges, is *triangulated* if each face is a triangle, formed by either three vertices or by one crossing and two vertices. Clearly, a triangulated 1-plane graph is 2-connected. The next observation follows from the definition of a triangulated 1-plane graph.

Observation 1 *Let G be a triangulated 1-plane graph. Every pair of crossing edges of G forms an empty kite, except for at most one pair of crossing edges if their crossing point is on the outer face of G .*

3 1-bend RAC Drawings of 1-planar Graphs

To prove Theorem 1 we give an algorithm that takes as input a simple 1-plane graph G with n vertices (see, e.g., Fig. 1(a)), and computes a 1-bend 1-planar RAC drawing Γ of G in $O(n)$ time. We assume that G is connected, as otherwise we can draw independently each connected component. The high-level idea is as follows. Augment G and modify its embedding to get a triangulated 1-plane graph, possibly containing parallel edges. Execute a suitable decomposition of the triangulated 1-plane graph and apply a recursive technique that computes a 1-bend 1-planar RAC drawing. Remove dummy vertices and edges.

Augmentation. The first step of the algorithm transforms G into a triangulated 1-plane graph G^+ by adding edges and vertices. The 1-planar embedding of G^+ may be different from that of G for the common part. Let (a, c) and (b, d) be two edges of G that cross in a point p . Let $\{a, b, c, d\}$ be the circular order of the vertices around p . For each such pair of crossing edges, we add an edge (a, b) , and draw³ it such that it follows the curves (a, p) and (p, b) . Similarly, we draw the three edges (b, c) , (c, d) and (d, a) . This operation ensures that each pair of crossing edges forms an empty kite. Also, this operation does not introduce edge crossings but it may create parallel edges. We denote by G_1 the resulting (multi)graph. For each pair of parallel edges e and e' of G_1 , such that $e \in G$ and $e' \in G_1$, we remove e from G_1 . This immediately implies that no parallel edge is crossed in G_1 . We then remove one edge for each pair of parallel edges e_1 and e_2 such that the curve $e_1 \cup e_2$ does not contain any vertex in its interior. We let G_2 be the resulting graph, which can be easily computed in $O(n)$ time, since G has $O(n)$ crossings (see, e.g., [36]). Figure 1(b) shows the graph G_2 obtained from the graph G of Fig. 1(a). We remark that a similar operation has been used by Alam et al. [2] in order to compute a straight-line drawable 1-planar embedding of a 3-connected 1-planar graph. However, only 3-connected graphs are considered by Alam et al., and in this case the augmented graph does not contain parallel edges [2]. We do not have any restriction on the connectivity of G , which poses additional issues in the construction and in the drawing of a suitable 1-planar embedding. We transform G_2 into a triangulated 1-plane graph. Note that a face of degree two consists of two parallel edges, thus only the outer face of G_2 can have degree two. In this case, each of the two parallel edges is part of an empty kite. Thus, we remove one of these two edges to make the degree of the outer face equal to three (it will be formed by two vertices and one crossing). Let f be an inner face of G_2 that is not a triangle. Such a face contains no crossings on its boundary, since each crossing is shared by exactly four triangular faces by the empty kite property. We add an *extra vertex* v_f inside f and connect it to all vertices (with multiplicity) on the boundary of f . Figure 1(c) shows the graph G^+ obtained from the graph G_2 in Fig. 1(b), extra vertices are drawn as squares. Since G_2 has $O(n)$ faces, G^+ has $O(n)$ vertices and edges, and it is computed in $O(n)$ time. The next lemma follows from the above discussion.

Lemma 1. *Graph G^+ is a triangulated 1-plane (multi)graph.*

Decomposition. We define a decomposition of G^+ inspired by *SPQR*-trees [12], but simpler and more direct for our purposes. The next lemma can be proved.

Lemma 2. *Let G be a triangulated 1-plane (multi)graph and let $\{u, v\}$ be a separation pair of G . There exist two parallel edges e, e' incident to u and v such that $\{u, v\}$ is not a separation pair for the graph obtained by removing from G all vertices inside the cycle $\{e, u, e', v\}$.*

³ For ease of description, here we are interpreting an embedding as a drawing.

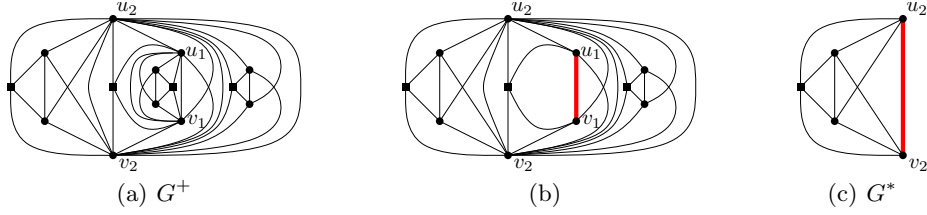


Fig. 2. Illustration for the decomposition step. Thick edges are thicker (and red).

By Lemma 2, for each separation pair $\{u, v\}$ of G^+ , there exist $k > 1$ parallel edges $\{e_1, \dots, e_k\}$ between u and v , such that the cycle $\{u, e_1, v, e_k\}$ encloses all other copies in its interior. We call the *inner graph* of (u, v) the subgraph G_{uv} of G^+ whose outer face is $\{u, e_1, v, e_k\}$, and an *inner component* of (u, v) each subgraph C_{uv}^i of G_{uv} whose outer face is $\{u, e_i, v, e_{i+1}\}$, for $i = 1, \dots, k - 1$. Let G_{uv} be an inner graph of G^+ that does not contain any inner graph as a subgraph. Replace G_{uv} with an edge between u and v , called *thick edge*; the resulting graph is still a triangulated 1-plane graph. Iterate this procedure until there are no more inner graphs to be replaced. This is done in $O(n)$ time and results in a simple triangulated 1-plane graph G^* , which is 3-connected by Lemma 2. Figure 2(c) shows the graph G^* obtained from the graph G^+ in Fig. 2(a), through the intermediate step in Fig. 2(b). The next lemma follows.

Lemma 3. *Graph G^* is a simple 3-connected triangulated 1-plane graph.*

Drawing. The overview of the drawing algorithm is as follows. Start with a 1-bend 1-planar RAC drawing of G^* , and then recursively replace thick edges with a 1-bend 1-planar RAC drawing of the corresponding inner graphs. Deleting the edges and vertices added by the augmentation step we get a 1-bend 1-planar RAC drawing of G . To compute a 1-bend 1-planar RAC drawing of G^* , first remove from G^* all pairs of crossing edges and denote by H^* the resulting plane graph (see Fig. 3(a)). Note that thick edges are never crossed by construction, and all faces of H^* have either degree 3 or degree 4. We can prove the following.

Lemma 4. *Graph H^* is 3-connected.*

Compute a planar straight-line drawing γ^* of H^* where all faces are strictly convex and the outer face is a prescribed polygon P ; this can be done by applying the linear-time algorithm by Chiba et al. [10] (see Fig. 3(b)). If the outer face of H^* has degree four, we let P be a trapezoid, else P is a triangle. Since all faces are either triangles or quadrangles, we can avoid three collinear vertices by slight perturbations (which cannot cause a face to become non convex). To reinsert the crossing edges, we distinguish between the inner faces and the outer face of H^* . Two crossing edges can be easily reinserted in an inner face, just drawing one of the two with no bend and the other with one bend, such that they cross at right angles (see, e.g., [3] and Fig. 3(c)). To reinsert two crossing edges e_1, e_2

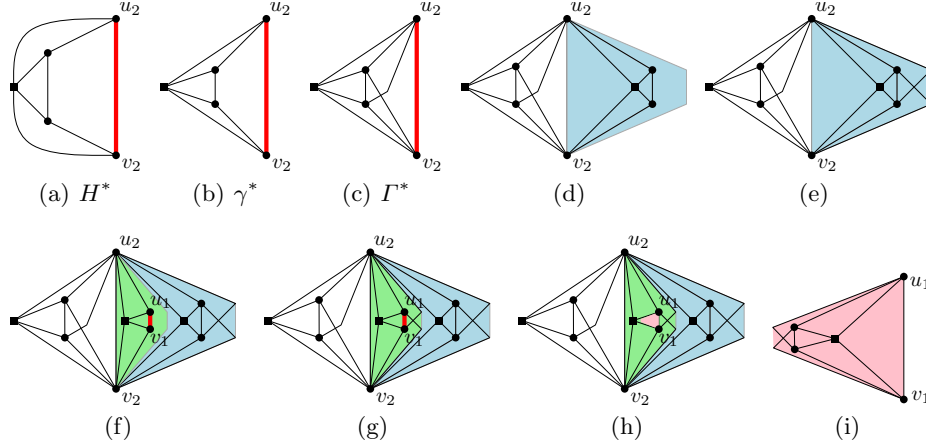


Fig. 3. Illustration for the drawing step.

in the outer face of H^* so that they form a right angle, we can draw e_1 and e_2 with one bend each. Namely, P is a trapezoid by construction. Assume that the minor base m and the greater base M of P are aligned with the horizontal axis. The first segment of e_1 is such that its rightmost endpoint p_1 coincides with the rightmost endpoint of m , and its leftmost endpoint q_1 is b units above the leftmost endpoint of m , where b is equal to the length of m . The second segment of e_1 has q_1 as rightmost endpoint, and its leftmost endpoint r_1 coincides with the leftmost endpoint of M . Edge e_2 is drawn symmetrically.

Consider now a thick edge (u, v) of G^* and its inner graph G_{uv} . Recall that G_{uv} consists of $k-1 \geq 1$ inner components C_1, \dots, C_{k-1} . Each C_i ($i = 1, \dots, k-1$) has two parallel edges e_i, e_{i+1} as outer face. Also, analogously to G^* , $C_i^- = C_i \setminus \{e_{i+1}\}$ is a simple 3-connected triangulated 1-plane graph (it is a subgraph of G^+ and all its inner graphs have been replaced by thick edges). Remove all crossing edges of C_{k-1}^- and let H_{k-1}^- be the resulting 3-connected plane graph. Compute a planar straight-line drawing γ_{k-1} of H_{k-1}^- such that all faces are strictly convex polygons and the outer face is a prescribed polygon P . If the outer face of H_{k-1}^- has degree three, P is a triangle whose side with corners u and v has length equal to the length of the thick edge (u, v) in Γ^* , and its height is small enough so that the thick edge (u, v) can be replaced with P without introducing crossings. If the outer face of H_{k-1}^- has degree four, P is a trapezoid such that its greater base has u and v as corners and the same length as the thick edge (u, v) in Γ^* . The height of P is such that the thick edge (u, v) can be replaced with P without introducing crossings. Also, the minor base of P is sufficiently short so that the pair of crossing edges on the outer face of H_{k-1}^- can be reinserted without introducing crossings in Γ^* , as described for H^* (see Fig. 3(d)). By the same argument used for H^* , all pairs of crossing edges can be reinserted so as to form right angle crossings and have at most one bend each

(see Fig. 3(e)). If $k-1 > 1$, we iterate this procedure and compute a drawing Γ_i^- for each C_i^- , for $i = k-2, \dots, 1$. The polygon representing the outer face of each Γ_i^- can be suitably chosen so to fit inside the face containing edge e_{i+1} of drawing of Γ_{i+1} . The union of all such drawings is a 1-bend 1-planar RAC drawing Γ_{uv} of G_{uv} (see Figs. 3(f) and 3(g)), with the exception of some parallel edges. Namely, the parallel edges e_1, \dots, e_k are represented by overlapping segments between u and v , and for our needs all of them but one can be removed from the drawing.

Repeat this procedure for each thick edge of G^* , and recursively apply the same technique for each inner graph of G^* ; see Figs. 3(h) and 3(i) for a complete illustration. The resulting drawing Γ is a 1-bend 1-planar RAC drawing of G^+ (except for some parallel edges). Removing dummy vertices and edges, we get the desired drawing of G . In terms of time complexity, each planar straight-line drawing with (strictly) convex faces is computed in linear time in the size of the input graph [10], and in linear time we can reinsert the crossing edges. Thus the whole procedure takes $O(n)$ time. This concludes the proof of Theorem 1.

4 Conclusions and Open Problems

We proved that every 1-planar graph admits a 1-planar RAC drawing with at most one bend per edge. The proof is constructive and based on a drawing algorithm, which may produce 1-bend 1-planar RAC drawings with exponential area: Is this area requirement necessary for some 1-planar graphs? Also, our algorithm may change the embedding of the input graph: Are there 1-planar embeddings that are not realizable as 1-bend RAC drawings? Characterizing straight-line 1-planar RAC drawable graphs is also an interesting problem.

References

1. Ackerman, E.: A note on 1-planar graphs. *Discrete Applied Mathematics* 175, 104–108 (2014), <http://dx.doi.org/10.1016/j.dam.2014.05.025>
2. Alam, M.J., Brandenburg, F.J., Kobourov, S.G.: Straight-line grid drawings of 3-connected 1-planar graphs. In: Wismath, S.K., Wolff, A. (eds.) *GD 2013*. LNCS, vol. 8242, pp. 83–94. Springer (2013)
3. Angelini, P., Cittadini, L., Didimo, W., Frati, F., Battista, G.D., Kaufmann, M., Symvonis, A.: On the perspectives opened by right angle crossing drawings. *J. Graph Algorithms Appl.* 15(1), 53–78 (2011)
4. Argyriou, E.N., Bekos, M.A., Symvonis, A.: Maximizing the total resolution of graphs. *Comput. J.* 56(7), 887–900 (2013)
5. Arikushi, K., Fulek, R., Keszegh, B., Morić, F., Tóth, C.D.: Graphs that admit right angle crossing drawings. *Comput. Geom.* 45(4), 169–177 (2012)
6. Bekos, M.A., van Dijk, T.C., Kindermann, P., Wolff, A.: Simultaneous drawing of planar graphs with right-angle crossings and few bends. *J. Graph Algorithms Appl.* 20(1), 133–158 (2016), <http://dx.doi.org/10.7155/jgaa.00388>
7. Biedl, T.C., Liotta, G., Montecchiani, F.: On visibility representations of non-planar graphs. In: Fekete, S.P., Lubiw, A. (eds.) *SoCG 2016*. LIPIcs, vol. 51, pp. 19:1–19:16. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik (2016), <http://www.dagstuhl.de/dagpub/978-3-95977-009-5>

8. Brandenburg, F.J.: 1-visibility representations of 1-planar graphs. *J. Graph Algorithms Appl.* 18(3), 421–438 (2014)
9. Brandenburg, F.J., Didimo, W., Evans, W.S., Kindermann, P., Liotta, G., Montecchiani, F.: Recognizing and drawing IC-planar graphs. *Theor. Comput. Sci.* 636, 1–16 (2016), <http://dx.doi.org/10.1016/j.tcs.2016.04.026>
10. Chiba, N., Yamanouchi, T., Nishizeki, T.: Linear algorithms for convex drawings of planar graphs. *Progress in graph theory* pp. 153–173 (1984)
11. Di Battista, G., Eades, P., Tamassia, R., Tollis, I.G.: *Graph Drawing*. Prentice Hall, Upper Saddle River, NJ (1999)
12. Di Battista, G., Tamassia, R.: On-line planarity testing. *SIAM J. Comput.* 25(5), 956–997 (1996)
13. Di Giacomo, E., Didimo, W., Eades, P., Liotta, G.: 2-layer right angle crossing drawings. *Algorithmica* 68(4), 954–997 (2014)
14. Di Giacomo, E., Didimo, W., Grilli, L., Liotta, G., Romeo, S.A.: Heuristics for the maximum 2-layer RAC subgraph problem. *Comput. J.* 58(5), 1085–1098 (2015)
15. Di Giacomo, E., Didimo, W., Liotta, G., Montecchiani, F.: Area requirement of graph drawings with few crossings per edge. *Comput. Geom.* 46(8), 909–916 (2013)
16. Didimo, W., Liotta, G., Mehrabi, S., Montecchiani, F.: 1-Bend RAC Drawings of 1-Planar Graphs. *ArXiv e-prints* abs/1608.08418 (2016), <http://arxiv.org/abs/1608.08418v1>
17. Didimo, W.: Density of straight-line 1-planar graph drawings. *Inf. Process. Lett.* 113(7), 236–240 (2013)
18. Didimo, W., Eades, P., Liotta, G.: Drawing graphs with right angle crossings. *Theor. Comput. Sci.* 412(39), 5156–5166 (2011)
19. Didimo, W., Liotta, G.: The crossing angle resolution in graph drawing. In: Pach, J. (ed.) *Thirty Essays on Geometric Graph Theory*. Springer (2012)
20. Didimo, W., Liotta, G., Romeo, S.A.: Topology-driven force-directed algorithms. In: Brandes, U., Cornelsen, S. (eds.) *GD 2010*. LNCS, vol. 6502, pp. 165–176. Springer (2010)
21. Didimo, W., Liotta, G., Romeo, S.A.: A graph drawing application to web site traffic analysis. *J. Graph Algorithms Appl.* 15(2), 229–251 (2011)
22. Eades, P., Liotta, G.: Right angle crossing graphs and 1-planarity. *Discrete Appl. Math.* 161(7-8), 961–969 (2013)
23. Grigoriev, A., Bodlaender, H.L.: Algorithms for graphs embeddable with few crossings per edge. *Algorithmica* 49(1), 1–11 (2007)
24. Hong, S., Eades, P., Liotta, G., Poon, S.: Fáry’s theorem for 1-planar graphs. In: Gudmundsson, J., Mestre, J., Viglas, T. (eds.) *COCOON 2012*. LNCS, vol. 7434, pp. 335–346. Springer (2012)
25. Huang, W.: Using eye tracking to investigate graph layout effects. In: *APVIS 2007*. pp. 97–100 (2007)
26. Huang, W., Eades, P., Hong, S.: Larger crossing angles make graphs easier to read. *J. Vis. Lang. Comput.* 25(4), 452–465 (2014)
27. Huang, W., Eades, P., Hong, S., Lin, C.: Improving force-directed graph drawings by making compromises between aesthetics. In: *VL/HCC*. pp. 176–183. IEEE (2010)
28. Huang, W., Hong, S.H., Eades, P.: Effects of crossing angles. In: *PacificVis 2008*. pp. 41–46. IEEE (2008)
29. Korzhik, V.P., Mohar, B.: Minimal obstructions for 1-immersions and hardness of 1-planarity testing. *J. Graph Theory* 72(1), 30–71 (2013)

30. Nguyen, Q., Eades, P., Hong, S., Huang, W.: Large crossing angles in circular layouts. In: Brandes, U., Cornelsen, S. (eds.) GD 2010. LNCS, vol. 6502, pp. 397–399. Springer (2010)
31. Pach, J., Tóth, G.: Graphs drawn with few crossings per edge. *Combinatorica* 17(3), 427–439 (1997)
32. Purchase, H.C.: Which aesthetic has the greatest effect on human understanding? In: Di Battista, G. (ed.) GD 1997. LNCS, vol. 1353, pp. 248–261. Springer (1997)
33. Purchase, H.C.: Effective information visualisation: a study of graph drawing aesthetics and algorithms. *Interacting with Computers* 13(2), 147–162 (2000)
34. Purchase, H.C., Carrington, D.A., Allder, J.: Empirical evaluation of aesthetics-based graph layout. *Empirical Software Engineering* 7(3), 233–255 (2002)
35. Ringel, G.: Ein Sechsfarbenproblem auf der Kugel. *Abh. Math. Semin. Univ. Hambg.* 29(1-2), 107–117 (1965)
36. Suzuki, Y.: Re-embeddings of maximum 1-planar graphs. *SIAM J. Discrete Math.* 24(4), 1527–1540 (2010)
37. Thomassen, C.: Rectilinear drawings of graphs. *J. Graph Theory* 12(3), 335–341 (1988)
38. Ware, C., Purchase, H.C., Colpoys, L., McGill, M.: Cognitive measurements of graph aesthetics. *Information Visualization* 1(2), 103–110 (2002)