

A $(7/2)$ -Approximation Algorithm for Guarding Orthogonal Art Galleries with Sliding Cameras

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Abstract. Consider a sliding camera that travels back and forth along an orthogonal line segment s inside an orthogonal polygon P with n vertices. The camera can see a point p inside P if and only if there exists a line segment containing p that crosses s at a right angle and is completely contained in P . In the minimum sliding cameras (MSC) problem, the objective is to guard P with the minimum number of sliding cameras. In this paper, we give an $O(n^{5/2})$ -time $(7/2)$ -approximation algorithm to the MSC problem on any simple orthogonal polygon with n vertices, answering a question posed by Katz and Morgenstern (2011). To the best of our knowledge, this is the first constant-factor approximation algorithm for this problem.

1 Introduction

In the classical art gallery problem, we are given a polygon and the objective is to cover the polygon with the union of visibility regions of a set of point guards while minimizing the number of guards. The problem was introduced by Klee in 1973 [13]. Two years later, Chvátal [2] showed that $\lfloor n/3 \rfloor$ point guards are always sufficient and sometimes necessary to guard the polygon. The orthogonal art gallery problem was first studied by Kahn et al. [4] who proved that $\lfloor n/4 \rfloor$ guards are always sufficient and sometimes necessary to guard the interior of a simple orthogonal polygon. Lee and Lin [9] showed that the problem of guarding a simple polygon using the minimum number of guards is NP-hard. Moreover, the problem was also shown to be NP-hard for orthogonal polygons [14]. Since then,

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the problem and its many variants have been studied extensively for different types of polygons (e.g., orthogonal polygons [14] and polyominoes [1]), different types of guards (e.g., points and line segments) and different visibility types. See the surveys by O'Rourke [13] or Urrutia [15] for a detailed history of the art gallery problem.

Recently, Katz and Morgenstern [6] introduced a variant of the art gallery problem in which *sliding cameras* are used to guard an orthogonal polygon. Let P be an orthogonal polygon with n vertices. A sliding camera travels back and forth along an orthogonal line segment s inside P . The camera can see a point $p \in P$ if there is a point $q \in s$ such that \overline{pq} is a line segment normal to s that is completely inside P . In the minimum sliding cameras (**MSC**) problem, the objective is to guard P using the minimum number of sliding cameras.

In this paper, we give an $O(n^{5/2})$ -time $(7/2)$ -approximation algorithm to the minimum sliding cameras (MSC) problem on any simple orthogonal polygon. To do this, we introduce the *minimum guarded sliding cameras (MGSC) problem*. In the MGSC problem, the objective is to guard P using a set of minimum cardinality of guarded sliding cameras. A sliding camera s is guarded by a sliding camera s' if every point on s is seen by some point on s' . Note that s and s' could be perpendicular, in which case s' and s mutually guard each other if and only if they intersect. If s and s' mutually guard each other and have the same orientation (e.g., both are horizontal), then the visibility region of s is identical to that of s' . Consequently, when minimizing the number of sliding cameras in the MGSC problem, it suffices to consider solutions in which each horizontal sliding camera is guarded by a vertical sliding camera and vice-versa. We first establish a connection between the MGSC problem and a related guarding problem on grids.

A grid D is a connected union of vertical and horizontal line segments; each maximal line segment in the grid is called a *grid segment*. A *point guard* x in grid D is a point that sees a point y in the grid if the line segment $xy \subset D$. Moreover, a *sliding camera* $p \in D$ is a point guard that moves along a grid segment $s \in D$. The camera p can see a point q on the grid if and only if there exists a point $p' \in s$ such that the line segment $\overline{p'q} \subset D$; that is, point q is seen by camera p if either q is located on s or q belongs to a grid segment that intersects s . Note that sliding cameras are called *mobile guards* in grid guarding problems [7, 8]. A *guarded* set of point guards and a guarded set of sliding cameras on grids are defined analogously to a guarded set of sliding cameras in polygons. A simple grid is defined as follows:

Definition 1 (Kosowski et al. [8]). *A grid is simple if (i) the endpoints of all of its segments lie on the outer face of the planar subdivision induced by the grid, and (ii) there exists an $\epsilon > 0$ such that every grid segment can be extended by ϵ in both directions such that its new endpoints are still on the outer face.*

Throughout the paper, we denote a simple orthogonal polygon by P ; note that the polygon P is a closed region. The rest of the paper is organized as follows. Section 2 presents related work. In Section 3, we give our $(7/2)$ -approximation algorithm to the MSC problem and we conclude the paper in Section 4.

2 Related Work and Definitions

The minimum sliding cameras problem was introduced by Katz and Morgenstern [6]. They first considered a restricted version of the problem in which only vertical cameras are allowed; by reducing the problem to the minimum clique cover problem on chordal graphs, they solved the problem exactly in polynomial time. For the generalized case, where both vertical and horizontal cameras are allowed, they gave a 2-approximation algorithm for the MSC problem under the assumption that the polygon P is x -monotone. Durocher and Mehrabi [3] showed that the MSC problem is NP-hard when the polygon P is allowed to have holes. They also gave an exact algorithm that solves in polynomial time a variant of the MSC problem in which the objective is to minimize the sum of the lengths of line segments along which cameras travel.

The guard problem on grids was first formulated by Ntafos [12]. He proved that a set of (stationary) point guards of minimum cardinality covering a grid of n grid segments has nm guards, where m is the size of the maximum matching in the intersection graph of the grid that can be found in $O(n^{5/2})$ time. Malafiejski and Zylinski [10] showed that the problem of finding a minimum-cardinality set of guarded point guards for a grid is NP-hard. Katz et al. [5] showed that the problem of finding a minimum number of sliding cameras covering a grid is NP-hard. Moreover, Kosowski et al. [7] proved that the problem of finding the minimum number of guarded sliding cameras covering a grid (we call this problem the **MMGG** problem) is NP-hard. Due to these hardness results, Kosowski et al. [8] studied the MMGG problem on some restricted classes of grids. In particular, they show the following result on simple grids:

Theorem 1 (Kosowski et al. [8].). *There exists an $O(n^2)$ -time algorithm for solving the MMGG problem on simple grids, where n is the number of grid segments.*

Throughout the paper, we denote optimal solutions for the MSC problem and the MGSC problem on P by OPT_P and OPT_{GP} , respectively. We denote the set of reflex vertices of P by $V(P)$ and let H_u and V_u be the maximum-length horizontal and vertical line segments, respectively, inside P through a vertex $u \in V(P)$. Let $L(P) = \{H_u \mid u \in V(P)\} \cup \{V_u \mid u \in V(P)\}$. Let L and L' be two orthogonal line segments (with respect to P) inside P ; the *visibility region* of L is the union of the points in P that are seen by the sliding camera that travels along L . Moreover, we say that L dominates L' if the visibility region of L' is a subset of that of L .

3 A $(7/2)$ -Approximation Algorithm for the MSC Problem

In this section, we present an $O(n^{5/2})$ -time $(7/2)$ -approximation algorithm for the MSC problem.

3.1 Relating the MGSC and MMGG problems

Consider an optimal solution X for the MSC problem and let X' be the multiset of line segments obtained by duplicating every line segment in X (i.e., each line segment of X occurs twice in X'). We observe that X' is a feasible solution for the MGSC problem and, therefore, we have the following observation.

Observation 1 *An optimal solution for the MGSC problem on P is a 2-approximation to an optimal solution for the MSC problem on P .*

We first consider how to apply a solution for the MGSC problem to the MSC problem. For the MGSC problem, the idea is to reduce the MGSC problem to the MMGG problem. Given any simple orthogonal polygon P , we construct a grid G_P associated with P as follows: initially, let G_P be the set of all line segments in $L(P)$. Now, for any pair of reflex vertices u and v where H_u dominates H_v (resp., V_u dominates V_v) in P , we remove H_v (resp., V_v) from G_P ; if two segments mutually dominate each other, remove one of the two arbitrarily. Let T_G be the set of remaining grid segments in G_P . Observe that G_P can be constructed in $O(n^2)$ time, where n is the number of vertices of P . We first show the following result:

Lemma 1. *Grid G_P is a simple and connected grid.*

Proof. It is straightforward from the construction of G_P that both endpoints of each grid segment in T_G lie on the boundary of P ; this means that the endpoints of every grid segment in T_G lie on the outer face of G_P and, therefore, G_P is simple. To show that G_P is connected, we first observe that the grid induced by the line segments in $L(P)$ is connected. We now need to show that the grid remains connected after removing the set of grid segments that are dominated by other grid segments. Let $s \in L(P)$ be a grid segment that is removed from $L(P)$ (i.e., $s \notin T_G$). It is straightforward to see that the set of grid segments that are intersected by s are also intersected by $s' \in T_G$, where s' is the grid segment that dominates s . Therefore, grid G_P is connected. \square

The objective is to solve the MMGG problem on G_P exactly and to use the solution S , the set of guarded grid segments computed, as the solution to the MGSC problem. However, S is not always a feasible solution to the MGSC problem since some regions in P might remain unguarded; see Figure 1 for an example. In the following, we characterize the regions of P that may remain unguarded by the line segments in S ; we call these the *critical regions* of P .

Consider S and choose any unguarded point p inside P . Let R_p be a maximal axis-aligned rectangle contained in P that covers p and is also not guarded by the line segments in S . We observe that (i) some line segments in T_G can guard R_p , and that (ii) no such line segments are in S since R_p is unguarded. Consider the maximal regions in P that lie immediately above, below, left, and right of R_p ; any sliding camera that sees any part of R_p must intersect one of these regions. See Figure 2(a) for an example; note that the hatched region cannot contain any line segment in S since R_p is unguarded. Moreover, the hatched region must contain

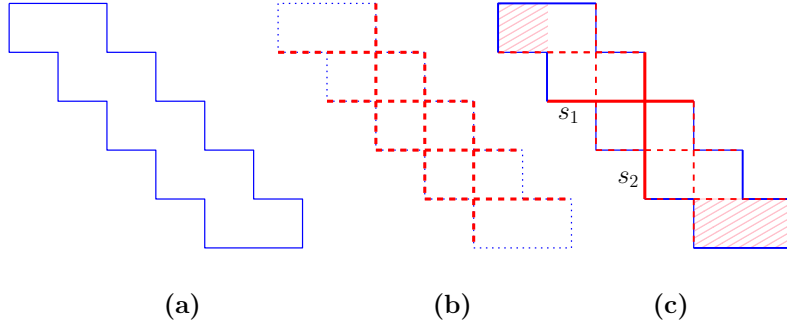


Fig. 1: (a) A simple orthogonal polygon P . (b) Grid G_P with set T_G of grid segments shown in red. (c) The set $S = \{s_1, s_2\}$ (represented by solid red line segments) is an optimal solution for the MMGG problem on G_P , but s_1 and s_2 cannot guard P entirely; in particular, the hatched regions of P are not guarded.

at least one line segment of T_G in both horizontal and vertical directions and without loss of generality we can assume that the length of these line segments is maximal (i.e., both endpoints are on the boundary of P).

The rectangle R_p defines a partition of P into three parts: the *vertical slab* through R_p , (i.e., the slab whose sides are aligned with the vertical sides of R_p), the subpolygon of P to the left of the vertical slab and the subpolygon of P to the right of the vertical slab. Similarly, another partition of P can be obtained by considering the *horizontal slab* through R_p ; see Figure 2(b) for an example. We know that the union of the visibility regions of the line segments in S is a connected subregion of the plane. Therefore, the set S can only be found on one side of each of the partitions of P and so S must be in one *corner* of the partitioned polygon. Without loss of generality, assume that S is on the bottom left corner of the partitioned polygon (see Figure 2(b)).

Let $\bar{S} \subseteq T_G$ be the set of line segments that can see R_p . Note that \bar{S} is non-empty. This is because \bar{S} is a subset of T_G and the line segments in \bar{S} can see R_p ; but R_p is a subregion of the polygon P . Since line segments in T_G guard the polygon entirely, there has to be at least one line segment in \bar{S} . Therefore, polygon P is partitioned into three subpolygons (see Figure 2(c)): the lower-left corner that is the location of S denoted by P_S , the lower-right and upper-left corners that correspond to line segments in \bar{S} denoted by $P_{\bar{S}}$, and the upper-right corner of the polygon that is unguarded denoted by P_U . Each line segment in \bar{S} intersects at least one line segment in S since the line segments in \bar{S} are not in S and S is feasible solution for the MMGG problem (see Figure 2(c)).

Lemma 2. *No line segment in T_G that is orthogonal to a line segment in \bar{S} can intersect P_U .*

Proof. To derive a contradiction, suppose without loss of generality that there is one such vertical line segment s intersecting P_U (as shown in Figure 2(c)). Since S is connected and is a feasible solution for the MMGG problem, there

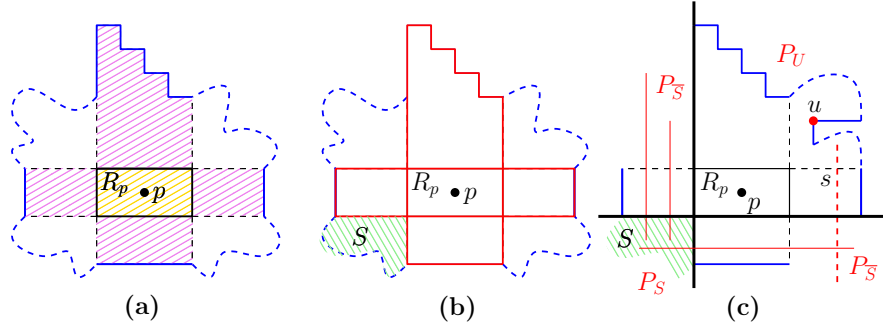


Fig. 2: (a) Point p inside the polygon P with rectangle R_p hatched in gold. The purple hatched region indicates the subregion of P covered by growing R_p orthogonally towards the boundaries of P ; the boundary of P is shown in blue. (b) The horizontal rectangle and the vertical histogram shown in red indicate, respectively, the horizontal and vertical slabs of the partitions induced by rectangle R_p . The hatched region of P on the bottom left corner of the partition indicates the location of the set S . (c) The partition of P into three subpolygons P_S , $P_{\bar{S}}$ and P_U . The line segments in \bar{S} (i.e., the set of line segments that can see R_p) are shown in red; observe that each line segment in \bar{S} intersects at least one line segment in S . The line segment s illustrates Lemma 2 and vertex u is in support of Lemma 3.

must be a line segment in S that guards s . So, S must contain a line segment in $P_{\bar{S}}$, which is a contradiction. \square

By Lemma 2, we conclude that R_p is guarded by the line segments in \bar{S} , but not by any line segment in S , and, furthermore, S is restricted to a corner as described above (i.e., the subpolygon P_S). We now show that each of the regions of P that are not guarded by the line segments in S must be a staircase with the reflex vertices oriented towards S .

To derive a contradiction, suppose that there exists a reflex vertex u (as shown in Figure 2(c)) in the unguarded subpolygon P_U of P . However, no line segment in T_G can intersect P_U by Lemma 2. This contradicts the existence of u . Therefore, any unguarded region of P by S must be bounded by the line segments in $L(P)$ ¹ on adjacent horizontal and vertical sides, and by a staircase of P on the other sides; we call these regions the *critical regions* of P , and denote R_C to be the set of critical regions of P . We now have the following lemma.

Lemma 3. *Every point of P that is not inside a critical region of P is visible to at least one line segment in S . Moreover, each critical region of P is a staircase.*

Let OPT_{GG} denote an optimal solution for the MMGG problem on G_P . We first prove that $|OPT_{GG}| \leq |OPT_{GP}|$.

¹ If the bounding line segment is not in T_G then a dominating line segment must be in T_G . See Figure 4 for an example.

Lemma 4. *For any feasible solution M for the MGSC problem on P , there exists a feasible solution S' for the MMGG problem on G_P such that $|S'| \leq |M|$.*

Proof. Let M be a feasible solution to the MGSC problem on P ; that is, M is a guarded set of orthogonal line segments inside P that collectively guard P . We construct a feasible solution S' for the MMGG problem such that $|S'| \leq |M|$. To compute S' , for each horizontal line segment $s \in M$ (resp., vertical line segment $s \in M$), move s up or down (resp., to the left or to the right) until it is collinear with a line segment $s \in L(P)$. If $s \in T_G$, then add s to S' ; otherwise, add s' to S' , where $s' \in L(P)$ is the line segment that dominates s . Note that there exists at least one such line segment s' because otherwise the line segment s would have not been removed from $L(P)$. It is straightforward to see that the union of visibility regions of line segments in M is a subset of the visibility regions of line segments in S' . Since the camera travelling along each line segment in M is seen by at least one other camera (see the definition of a guarded set of sliding cameras) and the grid G_P is entirely contained in P , we conclude that S' is a feasible solution for the MMGG problem on G_P . The inequality $|S'| \leq |M|$ follows from the fact that each line segment in M corresponds to at most one line segment in S' . This completes the proof of the lemma. \square

Next we need to find a set of minimum cardinality of orthogonal line segments inside P that collectively guard the critical regions of P .

3.2 Guarding Critical Regions: A (3/2)-Approximation Algorithm

In this section, we give an approximation algorithm for the problem of guarding the critical regions of P . The algorithm relies on reducing the problem to the minimum edge cover problem in graphs. The minimum edge cover problem in graphs is solvable in $O(n^{5/2})$ time, where n is the number of graph vertices [11]. Recall R_C , the set of critical regions of P . We first need the following result:

Lemma 5. *Every critical region in R_C is guarded entirely by some line segment in $L(P)$.*

Proof. Observe that if P is a rectangle, then the MSC problem is trivial to solve. Suppose that P is not a rectangle and so it has at least one reflex vertex. Furthermore, suppose that some regions of P are not guarded by S (the set of segments returned by solving the MMGG problem on G_P), i.e., the set R_C of critical regions of P is non-empty. Let $R \in R_C$ be a critical region of P . The lemma is implied by the fact that there exists at least one reflex vertex on the boundary of R ; this is because P is not a rectangle and the set of line segments in S do not guard R . It is now straightforward to see that one of the orthogonal line segments in $L(P)$ that passes through either the lowest or the highest reflex vertex of R can see the critical region R entirely. \square

We construct a graph H_P associated with P as follows: for each critical region $R \in R_C$, we add a vertex v_R to H_P . Two vertices v_R and $v_{R'}$ are adjacent in H_P

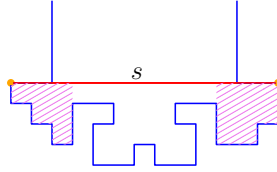


Fig. 3: Any orthogonal line segment inside P can guard at most two critical regions of P entirely.

if and only if there exists an orthogonal line segment inside P that can guard both critical regions R and R' entirely. Finally, we add a self-loop edge for every isolated vertex of H_P .

Lemma 6. *Any orthogonal line segment inside P can guard at most two critical regions of P entirely.*

Proof. Let s be an orthogonal line segment inside P . Observe that since each critical region of R is a staircase, line segment s must hit the boundary of R in order to guard R entirely. That is, the only way for s to guard R entirely is that at least one of its endpoints lies on the boundary of P , covering one entire edge of R ; see Figure 3 for an example. Therefore, s can guard at most two critical regions of P . \square

Lemma 7. *The problem of guarding the critical regions of P using only those line segments that may individually guard a critical region reduces to the minimum edge cover problem on H_P .*

Proof. We prove that (i) for any solution S to the minimum edge cover problem on H_P , there exists a solution S' for guarding the critical regions of P such that $|S'| = |S|$, and that (ii) for any solution S' to the problem of guarding the critical regions of P , there exists a solution S to the minimum edge cover problem on H_P such that $|S| = |S'|$.

Part 1. Choose any edge cover S of H_P . We construct a solution S' for guarding the critical regions of P as follows. For each edge $e = (v_R, v_{R'}) \in S$ let s_e be the line segment in P that can see both critical regions R and R' of P ; we add s_e to S' . It is straightforward to see that the line segments in S' collectively guard all critical regions of P .

Part 2. Choose any solution S' for guarding the critical regions of P . We now construct a solution S for the minimum edge cover problem on H_P . By Lemma 6, we know that every line segment in S' can see at most two critical regions of P . First, for each line segment in S' that can see exactly one critical region R of P , we add the self-loop edge of H_P that corresponds to R in S . Next, for each line segment $s \in S'$ that can see two critical regions of P , we add to S' the edge

in H_P that corresponds to s . Since any line segment in S' can see at most two critical regions of P , we conclude that every vertex of H_P is incident to at least one edge in S and, therefore, S is a feasible solution for the minimum edge cover problem on H_P . \square

In general, it is possible for the solution S' to be non-optimal. Only those edges which may individually guard a critical region were considered, while an optimal guarding solution may use two line segments to collectively guard a critical region, as shown in Figure 4. By Lemma 5, S' requires at most one edge for each critical region and, therefore, the number of guards returned by our algorithm is at most equal to the number of critical regions. If an optimal solution uses two segments to collectively guard one critical region, these two edges suffice to guard three critical regions, while our solution uses three segments to guard the same three critical regions. This results in an approximation factor of $3/2$ in the number of segments used to guard the set of critical regions.

We now examine the running time of the algorithm. Let n denote the number of vertices of P . To compute the critical regions of P , we first compute the set of staircases of P and, for each of them, we check to see whether they are guarded by the set of line segments in S . Each critical region of P can be found in $O(n)$ time and so the set of unguarded critical regions of P is easily computed in $O(n^2)$ time. Moreover, the graph H_P can be constructed in $O(n^2)$ time by checking whether there is an edge between every pair of vertices of the graph. Therefore, by Lemma 7 and the fact that the minimum edge cover problem is solvable on a graph with n vertices in $O(n^{5/2})$ time, we have the following lemma.

Lemma 8. *There exists a $(3/2)$ -approximation algorithm for solving the problem of guarding the critical regions of a simple orthogonal polygon P with n vertices in $O(n^{5/2})$ time.*

Given any simple orthogonal polygon P , we find a set of sliding cameras that guards P by first solving the instance G_P of the MMGG problem determined by P . This may leave a set of critical regions within P that remain unguarded. We then add a second set of sliding cameras to guard these critical regions. Recall the set S , an optimal solution to the MMGG problem on G_P , which can be found in $O(n^2)$ time, where n is the number of vertices of P . By Lemma 8, we approximate the problem of guarding the critical regions of P in $O(n^{5/2})$ time; let S_C be the solution returned by the algorithm. Since the union of the critical regions of P is a subset of P , any feasible solution to the MSC problem also guards the critical regions of P . Therefore, $|S_C| \leq (3/2) \cdot |OPT_P|$. Moreover, by Lemma 4 we know $|OPT_{GG}| \leq |OPT_{GP}|$ and since $|OPT_{GP}| \leq 2 \cdot |OPT_P|$ we have that $|OPT_{GG}| \leq 2 \cdot |OPT_P|$. Therefore, by combining S and S_C we obtain a feasible solution to the MSC problem whose cardinality is at most $7/2$ times $|OPT_P|$; that is, $|S \cup S_C| \leq (7/2) \cdot |OPT_P|$. This gives the main result:

Theorem 2. *There exists an $O(n^{5/2})$ -time $(7/2)$ -approximation algorithm for the MSC problem on any simple orthogonal polygon P with n vertices.*

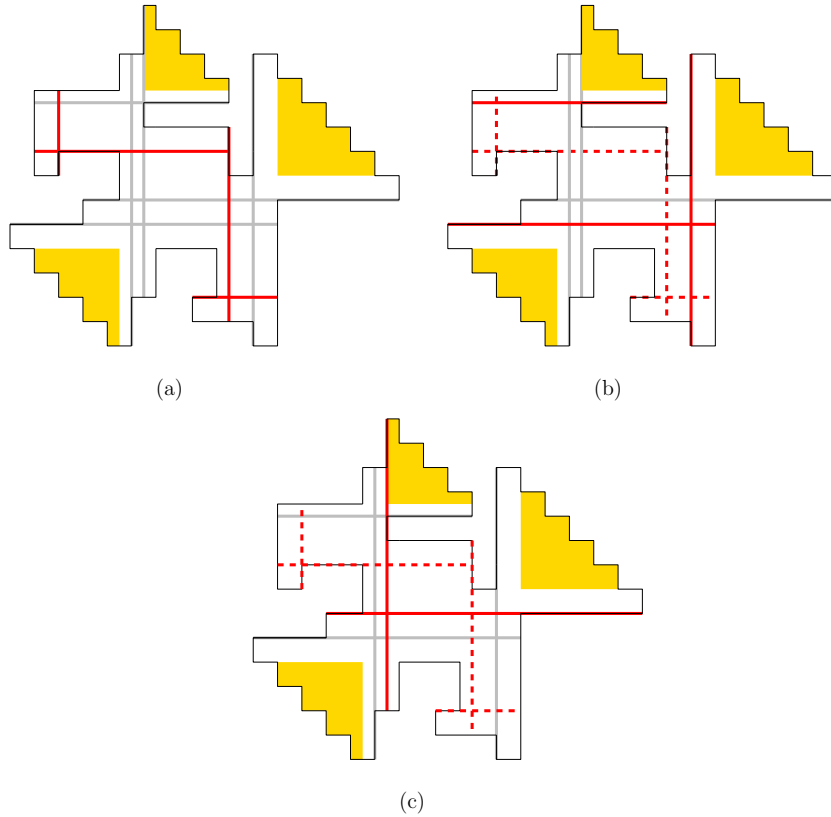


Fig. 4: An example of a polygon for which two line segments may collectively guard a critical region. (a) The polygon has unguarded critical regions after finding an optimal MMGG solution. The thick red line segments indicate guarding lines, light gray line segments are unused, and the shaded regions are unguarded. (b) A possible solution (solid red line segments) for guarding the critical regions using the edge guarding approach. (c) An optimal critical region guarding solution for the polygon.

As a consequence of our main result, we note that $|S_C| \leq (3/2) \cdot |OPT_P| \leq (3/2) \cdot |OPT_{GP}|$ and again by Lemma 4 we know $|OPT_{GG}| \leq |OPT_{GP}|$. Therefore, $|S \cup S_C| \leq (5/2) \cdot |OPT_{GP}|$. To show that the set $S \cup S_C$ is a feasible solution for the MGSC problem, we first observe that every line segment in S is guarded by at least one other line segment in S . Moreover, every grid segment that is not in S is guarded by some line segment in S because S is a feasible solution for the MMGG problem. This means that every line segment in S_C is guarded by at least one line segment in S . Therefore, we have the following result.

Corollary 1. *Given a simple orthogonal polygon P with n vertices, there exists an $O(n^{5/2})$ -time $(5/2)$ -approximation algorithm for the MGSC problem on P .*

4 Conclusion

In this paper, we studied a variant of the art gallery problem, introduced by Katz and Morgenstern [6], where sliding cameras are used to guard an orthogonal polygon and the objective is to guard the polygon with minimum number of sliding cameras. We gave an $O(n^{5/2})$ -time $(7/2)$ -approximation algorithm to this problem by deriving a connection between a guarded variant of this problem (i.e., the MGSC problem) and the problem of guarding simple grids with sliding cameras. The complexity of the problem remains open for simple orthogonal polygons. Giving an α -approximation algorithm, for any $\alpha < 7/2$, is another direction for future work. Finally, studying the MGSC problem (the complexity of the problem or improved approximation results) might be of independent interest.

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