Erratum to: Computing Partitions of Rectilinear Polygons with Minimum Stabbing Number

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Abstract. In this note, we report an error in our paper "*Computing Partitions of Rectilinear Polygons with Minimum Stabbing Number*" [2]. Given an orthogonal polygon P and a partition of P into rectangles, the *stabbing number* of the partition is defined as the maximum number of rectangles stabbed by any orthogonal line segment inside P. Abam et al. [1] introduced the problem of finding a partition of P into rectangles with minimum stabbing number and gave a 3-approximation algorithm for this problem. We gave a 2-approximation algorithm for a *conforming* version of this problem based on a Linear Program (LP) formulation of the problem [2], and claimed that generalizing the LP will also result in a 2-approximation algorithm for the general problem studied by Abam et al. [1]. In this note, we give a counterexample showing that generalizing the LP may not always result in a 2-approximation algorithm for the general version of the problem.

1 Introduction

Let P be a simple orthogonal polygon and let R denote a partition of P into rectangles. A partition of R into rectangles is obtained by inserting orthogonal *edges* into the interior of the polygon such that edge's endpoint is either on the boundary of the polygon or in the interior of another edge. The *stabbing number* of R is defined as the maximum number of partition rectangles stabbed by any orthogonal line segment inside P. We study the problem of finding a partition of P into rectangles whose stabbing number is minimum over that of all such partitions of P. There are two types of partitions of P into rectangles: in a *conforming* partition of P both endpoints of every edge of the partition must be on the boundary of the polygon while in a *general* partition this constraint is relaxed; that is, the edges of the partition may end on the interior of each other. Throughout this paper, we call the corresponding problems the conforming and the general problems, respectively.

Abam et al. [1] introduced the general problem and gave a 3-approximation algorithm for this problem; their approximation result is based on a polynomial-time exact algorithm for solving the general problem for a histogram. Durocher and Mehrabi [2] introduced the conforming problem for which they gave a polynomial-time exact algorithm for histograms. They also gave a polynomial-time 2-approximation algorithm for the conforming problem for any orthogonal polygon based on a Linear Program (LP) formulation of the problem. In the following, we briefly describe the LP given by Durocher and Mehrabi [2].

We first observe that a partition divides polygon P into convex regions if and only if at least one of the edges of the partition is anchored at every reflex vertex of P. For each reflex vertex u of P, let H_u (resp., V_u) be the maximal horizontal (resp., vertical) line segment inside P that has one endpoint at u. Moreover, consider two binary variables u_h and u_v that are correspond to H_u and V_u, respectively. Since at least one of H_u and V_u has to be in the partition, we have $u_h + u_v \geq 1$ for every reflex vertex u of P. Moreover, we observe that if H_v intersects V_u for two reflex vertices u and v of P, then at most one of H_v and V_u can be in the partition. Therefore, the followings are the constraints of the Integer LP (ILP) for the conforming problem:

$$
\begin{aligned}
& u_h + u_v \geq 1, && \forall u \in V(P), \\
& v_h + u_v \leq 1, && \text{if } H_v \text{ intersects } V_u, \\
& u_h, u_v \in \{0, 1\}, && \forall u \in V(P).
\end{aligned} \tag{1}
$$

We omit the details of the objective function of the ILP (as it requires to introduce more notations) and we just describe the idea behind it: we select a set S of orthogonal line segments inside P such that, for a given partition of P, the stabbing number of the partition is determined by the number of rectangles intersected by the line segments in S; that is, the line segment in S that intersects the maximum number of rectangles of the partition determines the stabbing number of the partition. We next relax the constraint (1) to $u_h, u_v \geq 0, \forall u \in V(P)$ to get an LP for the conforming problem. Let s^* be an optimal solution to the LP. We use the following rounding to get a 2-approximation to an optimal solution for the conforming problem. For each vertex $u \in V(P)$:

$$
u_h = \begin{cases} 0, & \text{if } s^*(u_h) \leq 1/2, \\ 1, & \text{if } s^*(u_h) > 1/2, \end{cases} \quad \text{and} \quad u_v = \begin{cases} 0, & \text{if } s^*(u_v) < 1/2, \\ 1, & \text{if } s^*(u_v) \geq 1/2. \end{cases} \tag{2}
$$

We also claimed that a generalization of this LP and using the same rounding method will result in a polynomial-time 2-approximation algorithm for the general problem. In Section 2, we present the details of generalizing the LP and show that it may not result in a 2-approximation algorithm.

2 The Generalized LP

In this section, we first present the generalized LP for general problem given by Durocher and Mehrabi [2] and then will note that it does not result in a 2-approximation algorithm for the general problem. In the following, we present the details of how the LP described in Section 1 is generalized.

2.1 Generalizing the LP

The objective function of the LP is defined as follows. Let u be a vertex in $V(P)$. V_u intersects H_v, for zero or more reflex vertices v. Therefore, V_u is partitioned into a number of line segments whose union is V_u. Similarly, H_u intersects V_v, for zero or more reflex vertices v. Therefore, H_u is partitioned into a number of line segments whose union is H_u. Let $L(V_u) = \{s_{1u}, s_{2u}, \dots, s_{cu}\}$ denote the set of line segments of V_u in the order of visiting if one walks on V_u started from u and towards the edge of P opposite to u. Similarly, let $L(H_u) = \{s'_{1u}, s'_{2u}, \dots, s'_{c'u}\}$ denote the set of line segments of H_u in the order of visiting if one walks on H_u started from u and towards the edge of P opposite to u. For any vertex $u \in V(P)$, we call a line segment in $L(V_u) \cup L(H_u)$ a *fragment*. A binary variable u_{v_i} is associated with s_{iu}, for $1 \le i \le c$, such that u_{v_i}=1 if and only if s_{iu} is present in the partition. Similarly, a binary variable u_{h_i} is associated with s'_{iu}, for $1 \le i \le c'$, such that u_{h_i}=1 if and only if s'_{iu} is present in the partition.

Next, we define two variables $u_{\Sigma h}$ and $u_{\Sigma v}$ for each reflex vertex $u \in V(P)$. Let S_u (resp. S'_u) be the set of fragments that are crossed by a maximal horizontal (resp. vertical) line segment that passes through u and is completely contained in P. Note that a maximal axis-parallel line segment may include a portion of the boundary of P. Then,

$$u_{\Sigma h} = 1 + \sum_{a \in S_u} var(a) \quad \text{(resp.,} \quad u_{\Sigma v} = 1 + \sum_{a \in S'_u} var(a)),$$

where $var(a)$ denotes the variable correspond to the fragment a. Therefore, the stabbing number of a partition corresponds to

$$1 + \max_{u \in V(P)} \{\max\{u_{\Sigma h}, u_{\Sigma v}\}\}. \tag{3}$$

Similar to Section 1, the LP formulation is computed from an ILP. The constraints of the ILP is computed as follows that along with (3) complete the ILP formulation of the problem. First, it is easy to see that the following constraints must be followed. For every vertex $u \in V(P)$ with $|L(V_u)| = c$ and $|L(H_u)| = c'$:

$$\begin{aligned} u_{v_1} + u_{h_1} &\ge 1, \\ u_{v_1} \ge u_{v_2} &\ge \cdots \ge u_{v_c}, \\ u_{h_1} \ge u_{h_2} &\ge \cdots \ge u_{h_{c'}}, \\ u_{v_i}, u_{h_j} \in \{0,1\}, 1 \le i &\le c, 1 \le j \le c'. \end{aligned} \tag{4}$$

Now, let $u \in V(P)$ be as described above and, moreover, let v be a vertex in $V(P)$ such that $|L(V_v)| = t$ and $|L(H_v)| = t'$. Now, suppose that s'_{iu}, where $1 \le i < c'$, intersects s_{j_v} in a partition, where $1 \le j < t$.[1] Then, exactly one of $s'_{(i+1)_u}$ and $s_{(j+1)_v}$ must be present in the partition. More precisely, (i) if s'_{iu} is

[1] It is not possible that s'_{iu} intersects s_{tv} for $1 \le i < c'$, or, s_{j_v} intersects $s'_{c'u}$, for $1 \le j < t$.

present in a partition, then exactly one of $s'_{(i+1)_u}$ and $s_{(j+1)_v}$ must be present in the partition and, similarly, (ii) if s_{j_v} is present in a partition, then exactly one of $s_{(j+1)_v}$ and $s'_{(i+1)_u}$ must be present in the partition. . By (i):

$$\begin{aligned} &u_{h_i} \rightarrow [(u_{h_{i+1}} \wedge \sim v_{v_{j+1}}) \vee (\sim u_{h_{i+1}} \wedge v_{v_{j+1}})] \\ &\Rightarrow 1 - u_{h_i} + 1 - v_{v_{j+1}} + 1 - u_{h_{i+1}} \geq 1, \text{ and } 1 - u_{h_i} + u_{h_{i+1}} + v_{v_{j+1}} \geq 1. \end{aligned} \tag{5}$$

Similarly, by (ii):

$$\begin{aligned} &v_{v_j} \rightarrow [(v_{v_{j+1}} \wedge \sim u_{h_{i+1}}) \vee (\sim v_{v_{j+1}} \wedge u_{h_{i+1}})] \\ &\Rightarrow 1 - v_{v_j} + 1 - v_{v_{j+1}} + 1 - u_{h_{i+1}} \geq 1, \text{ and } 1 - v_{v_j} + u_{h_{i+1}} + v_{v_{j+1}} \geq 1. \end{aligned} \tag{6}$$

Finally, an LP is computed from the ILP by replacing the last constraint of (4) with $u_{v_i}, u_{h_j} \geq 0$, where $1 \leq i \leq c, 1 \leq j \leq c'$:

$$\begin{aligned} &\text{minimize (3)} && (7) \\ &\text{subject to } \forall u \in V(P): \\ &\quad u_{v_1} + u_{h_1} \geq 1, && (8) \\ &\quad u_{v_1} \geq u_{v_2} \geq \cdots \geq u_{v_c} && (9) \\ &\quad u_{h_1} \geq u_{h_2} \geq \cdots \geq u_{h_{c'}}, && (10) \\ &\quad u_{v_i}, u_{h_j} \geq 0, 1 \leq i \leq c, 1 \leq j \leq c'. && (11) \\ &\quad \forall u, v \in V(P) \text{ that } s'_{i_u} \text{ intersects } s_{j_v}: && (12) \\ &\quad 1 - u_{h_i} + 1 - v_{v_{j+1}} + 1 - u_{h_{i+1}} \geq 1, \\ &\quad 1 - u_{h_i} + u_{h_{i+1}} + v_{v_{j+1}} \geq 1, \\ &\quad 1 - v_{v_j} + 1 - v_{v_{j+1}} + 1 - u_{h_{i+1}} \geq 1, \\ &\quad 1 - v_{v_j} + u_{h_{i+1}} + v_{v_{j+1}} \geq 1. \end{aligned}$$

Let s^* be a solution to (7). A rounding method similar to the one described in Section 1 for s^* is used to obtain a feasible solution for the problem: for each vertex $u \in V(P)$ with $L(V_u) = \{s_{1_u}, s_{2_u}, \ldots, s_{c_u}\}$ and $L(H_u) = \{s'_{1_u}, s'_{2_u}, \ldots, s'_{c'_u}\}$:

$$u_{v_i} = \begin{cases} 0, & \text{if } s^*(u_{v_i}) < 1/2, \\ 1, & \text{if } s^*(u_{v_i}) \geq 1/2, \end{cases} \tag{13}$$

where $i = 1, 2, \cdots, c$, and

$$u_{h_j} = \begin{cases} 0, & \text{if } s^*(u_{h_j}) \leq 1/2, \\ 1, & \text{if } s^*(u_{h_j}) > 1/2, \end{cases} \tag{14}$$

where $j = 1, 2, \cdots, c'$. We claimed that similar arguments as for the conforming problem can be used to show that this rounding method results in a 2-approximation algorithm for the general version of the problem.

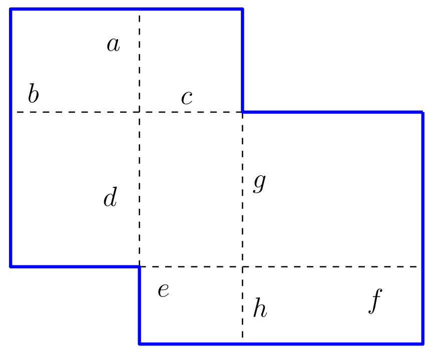

Fig. 1: A counterexample for the 2-approximation algorithm to the generalized version of the problem.

2.2 A Counterexample

In this section, we show that the generalized version of the LP as described above may not always result in a 2-approximation algorithm for the general problem. Consider the simple polygon shown in Figure 1; for simplicity, the fragments are labelled from a to h. The generalized LP for this polygon is as follows. By (4), we have:

$$d + e \geq 1, c + g \geq 1 \text{ and } c \geq b, d \geq a, g \geq h, e \geq f,$$

where $a, b, c, d, e, f, g, h \geq 0$. Moreover, by (5) and (6), we have:

$$1 - d + a + b \geq 1 \text{ and } 1 - d + 1 - a + 1 - b \geq 1$$

$$1 - c + b + a \geq 1 \text{ and } 1 - c + 1 - b + 1 - a \geq 1$$

$$1 - g + h + f \geq 1 \text{ and } 1 - g + 1 - h + 1 - f \geq 1$$

$$1 - e + f + h \geq 1 \text{ and } 1 - e + 1 - f + 1 - h \geq 1$$

It is straightforward to see that the assignment $a = 2/3 + \epsilon$, $b = 1/3 + \epsilon$, $c = 1/3 + \epsilon$, $d = 2/3 + \epsilon$, $e = 2/3$, $f = 1/3 + \epsilon$, $g = 2/3$ and $h = 1/3 + \epsilon$ is a feasible solution for this LP. By (13) and (14), we get $a = d = e = g = 1$ and $b = c = f = h = 0$, which does not result in a valid rectangular partition of the polygon shown in Figure 1.

3 Conclusion

In this note, we reported an error in our paper "*Computing Partitions of Rectilinear Polygons with Minimum Stabbing Number*" [2]. We showed by a couterexample that generalizing the LP given in [2] for the conforming problem does not always result in a 2-approximation algorithm for the general problem. We note that Breno and de Souza [3] have independently discovered a different counterexample. Thus, to the best of our knowlwedge, the 3-approximation algorithm of

Abam et al. [1] remains the best approximation algorithm for the general problem. Designing algorithms with better approximation factor or finding a suitable rounding method for our LP remain open.

References

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